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Mathematics. — “A Null System (1, 2, 3).” By Prof. JAN DE VRIES.

(Communicated at the meeting of February 24, 1923).

1. We consider as given a congruence $[\varrho^3]$ of twisted cubics with the base points $C_1, C_2, C_3, C_4, C_5^1)$ and the crossing straight lines a and b .

Through a point N there passes one curve ϱ^3 ; let r be the tangent at N and t the transversal of a and b through N . We conjugate $r \equiv rt$ to N as a *null plane*.

The curves ϱ^3 touching a plane r have their points of contact in a conic ϱ^2 . The transversal t lying in r , cuts ϱ^2 in the *null points* N_1 and N_2 of r .

If r revolves round the straight line l , t describes a scroll $(t)^2$ and ϱ^2 a cubic surface through l . The locus of N is accordingly a twisted curve λ^6 , which has evidently l , hence also a and b , as trisecants.

We have therefore a *null system* with the characteristic numbers $\alpha = 1, \beta = 2, \gamma = 3$.

2. The points C_k are *singular*; for C_k carries one straight line t but ∞^2 straight lines r . The null planes of C_k form a pencil of planes round t as axis.

Also the points A of a and B of b are *singular*. For each of them carries ∞^1 straight lines t which are combined to a plane pencil. The null planes of each of these points form a pencil of which the axis lies in the tangent r . These axes form two cubic scrolls $(r)^3$.

Other *singular points* S can only arise through coincidence of the straight lines t and r . Now the tangents of the curves ϱ^3 form a complex of the 6th order and this complex has a scroll $(n)^{12}$ in common with the bilinear congruence $[t]$. On each straight line n there lies a point S to which any plane through n corresponds as null plane.

As l is intersected by 12 straight lines n , the corresponding curve λ^6 contains 12 points S .

¹⁾ The principal properties of this congruence are to be found for instance in R. STURM: *Die Lehre von den geometrischen Verwandtschaften*, Part IV, p. 470.

3. The null points of the planes passing through the point P , lie on a surface $(P)^4$. For P is the null point of one definite plane of the sheaf and on a straight line l through P there lie the null points of three planes through l .

The intersection of the surface $(P)^4$ and $(Q)^4$ consists of the curve λ^6 corresponding to PQ , the straight lines a and b , and a curve σ^9 which is the locus of the *singular points* S and which passes evidently through the 5 base points C_k .

Three surfaces $(O)^4$, $(P)^4$ and $(Q)^4$ have in the first place the curve σ^9 in common. The points which they have further in common, are apparently the points of intersection of $(O)^4$ with the curve λ^6 corresponding to PQ . To them there belong the 12 points S on λ^6 and the 2×3 points A and B on λ^6 ; the remaining two are the null points of the plane OPQ .

4. Any plane α through a is *singular*; it contains a plane pencil (t) and each ray t cuts the conic φ^2 (§ 1) in two null points. Analogously the planes β through b are *singular*.

Also the ten planes σ each containing three base points C_i are *singular*. For in $\sigma_{1,2,3}$ there lies a pencil of conics of which each individual is combined with the straight line $C_4 C_5$ to a curve φ^3 ; they cut the straight line t in $\sigma_{1,2,3}$ in an involution of null points.

The surface $(P)^4$ contains the conics α^2 and β^2 lying in the planes Pa and Pb , and the intersection p of these planes. The straight line p is singular in this respect that it is a null ray for *each* of its points. The *singular null rays* p form the bilinear congruence with the director lines a and b .

Also the ten straight lines $C_k C_l$ are *singular*; for through each point on such a straight line r_{kl} there passes one straight line t , while r_{kl} may be considered as a tangent.

Mathematics. — “A Congruence (1,0) of Twisted Cubics”. By
Prof. JAN DE VRIES.

(Communicated at the meeting of February 24, 1923).

1. The twisted cubics through four points C_1, C_2, C_3, C_4 cutting the straight line b twice, form a linear congruence $[\varrho^3]$; for through any point there passes one ϱ^3 . The base points C are the *cardinal points*, b is a *cardinal chord*.

If d is a chord of one of the ϱ^3 , $d(C_1 C_2 C_3 C_4) = b(C_1 C_2 C_3 C_4)$. The chords d form therefore a *tetrahedral complex*; a ray l not belonging to this complex, is not cut twice by any ϱ^3 : the *class* of the congruence is *zero*.

Together with C_k and b a chord d defines a hyperboloid; on this there lie ∞^1 curves ϱ^3 and these define on d an involution; d is consequently a *tangent* to two curves.

The tangents meeting at a point P , lie on the complex cone of P ; their *points of contact* form a twisted curve of the 5th order, ϱ^6 , passing through P .

2. Let B_4 be the point of intersection of b with the plane $\gamma_{123} \equiv C_1 C_2 C_3$. Each conic ϱ^2 through the points C_1, C_2, C_3, B_4 is a component part of a degenerate ϱ^3 ; the transversal t_4 through C_4 resting on b and ϱ^2 is the second component part. The straight lines t_4 form the *pencil of rays* through C_4 in the plane $C_4 b$. There are therefore *four* pencils of rays formed by *singular straight lines*.

The pairs of lines of the pencil (ϱ^2) produce three figures each consisting of three straight lines, e.g. the combination of $C_1 C_2, C_3 B_4$ and the straight line t_4 resting on $C_1 C_2$. There are evidently *twelve* figures consisting of *three straight lines*.

3. With a view to finding the order of the surface A formed by the ϱ^3 cutting a straight line l , we determine the intersection of A with the plane γ_{123} . It consists of two conics of the pencil (ϱ^2); the former cuts l , the latter is a component part of the ϱ^3 which is defined by the transversal through C_4 of b and l . Hence A is a *surface* of the 4th order; the cardinal points C are apparently *double points* of A . A ϱ^3 not lying on A , can only cut this surface in the points C and on the cardinal chord b ; from this there follows that b is a *double straight line*.

On \mathcal{A}' there lie 9 straight lines and 8 conics.

The straight lines resting on b and l , determine a representation of \mathcal{A}' on a plane.

A straight line l_1 through a point C cuts \mathcal{A}' in two more points outside C ; from this follows that the ϱ^3 cutting l_1 , lie on a hyperboloid; this is entirely defined by l_1 , b and C_k . Analogously the ϱ^3 resting in a fixed point on b or on a straight line intersecting b , form respectively a quadric cone or a hyperboloid.

4. A plane λ through l cuts \mathcal{A}' along a curve λ^3 which has a double point on b . In each of the three points of intersection of λ^3 with l , λ is touched by a ϱ^3 . Hence the curves ϱ^3 touching a plane δ , have their *points of contact* on a curve δ^2 .

Let B be a point of b ; the ϱ^3 through the five points B and C_k touching δ , form a surface of the 10th order with sextuple points in B and C_k ¹⁾. There are accordingly 4 ϱ^3 through B and C_k which have b as a chord; consequently b is *quadruple* on the locus \mathcal{A} of the ϱ^3 touching the plane δ and belonging to the congruence (1,0). Also it appears that \mathcal{A} has *quadruple points* in C_k . Accordingly an arbitrary ϱ^3 of the (1,0) has 24 points in common with \mathcal{A} , i. e. \mathcal{A} is a *surface* of the 8th order.

5. \mathcal{A}^3 has the curve of contact δ^3 and a conic δ^2 in common with the plane δ . The curves δ^3 and δ^2 touch each other in 3 points; there are therefore *three* curves ϱ^3 which *osculate* the plane δ .

If \mathcal{A} revolves round l , δ^3 describes a surface of the fourth order with the single straight line l .

On the curve ϱ^3 cutting l in R , the pencil of planes (δ) defines an involution; l is therefore cut by two tangents of ϱ^3 . Consequently through l there pass two planes in which R is a point of the "complementary" curve δ^2 . Hence δ^2 describes a surface of the fourth order with the double straight line l .

Let us now consider the relation between the points P and Q which the curves δ^3 and δ^2 in a plane δ have in common with l . Through P there passes one ϱ^3 ; the tangent at P defines the plane δ , hence two points Q . Through Q there pass two ϱ^3 , hence two curves δ^2 , and two planes σ each containing a curve δ^3 ; six points P are therefore associated to Q . If two homologous points P and Q coincide, there arises a double coincidence of the (6,2), for at that point a ϱ^3 is osculated by the plane δ . On l there lie therefore four points N for which the plane of osculation ν passes through l .

¹⁾ This is easily seen from the intersection of this surface with γ_{123} , which consists of 2 conics and 3 double straight lines.

6. If we consider N as the null-point of ν , there arises a *null-system* with the characteristic numbers $\alpha = 1$, $\beta = 3$, $\gamma = 4$ (§ 5).

If ν continues to pass through a point P , the locus of N consists of a surface $(P)^5$ and the four pencils of rays round the points C_k in the planes $C_k b$ (§ 2).

If ν revolves round the straight line l , ν describes a curve λ' and the four singular rays through C_k which rest on l .

The surfaces $(P)^5$ and $(Q)^5$ have in common the curve λ' corresponding to PQ , and the 18 singular straight lines $C_k C_l$ and $C_k B_l$.

With a q^3 $(P)^5$ has in common the 3 points of which the planes of osculation pass through P ; the remaining 12 common points lie in the cardinal points C ; these are therefore triple points of $(P)^5$. The planes of osculation in C_k envelop accordingly a cone of the third class.

Mathematics. — “*A Representation of the Line Elements of a Plane on the Tangents of a Hyperboloid.*” By Prof. JAN DE VRIES.

(Communicated at the meeting of March 24, 1923).

1. In order to arrive at a representation of the line elements (P, l) of a plane α , we consider a hyperboloid H which touches α in A , and which cuts it along the straight lines a_1 and a_2 . Let R be the projection of P on H out of the point O of H , ϱ the tangent plane at R , r the intersection of ϱ with the plane Ol ; we consider r as the *image* of the *line element* formed by P and l .

If, inversely, r is a tangent of H , R the point of contact, P the projection of R , l the projection of r , the line element (P, l) has the tangent r for image¹).

We shall call the straight lines of H which cut each other in O , b_1 and b_2 ; b_1 cuts α in a point B_1 of a_2 , b_2 passes through a point B_2 of a_1 .

2. If l passes through B_1 and P coincides with B_1 , R is the point of contact of the plane $b_1 l$, and any tangent r lying in this plane, may be considered as the image (B_1, l) . Hence (B_1, l) is a *singular* element and its image is the plane pencil (r) round R . If l revolves round B_1 , the plane pencil (r) describes the parabolic bilinear congruence with the directrix b_1 , formed by the tangents which have their points of contact on b_1 . Analogously the line elements (B_2, l) are *singular*.

If B is an arbitrary point of the straight line $b \equiv B_1 B_2$, R lies in O . The line element (B, b) is therefore also *singular* and is represented by the plane pencil (o) of the straight lines that touch H in O and lie in the tangent plane ω .

Hence, inversely, any tangent o is *singular*, as it represents all elements (B, b) . But at the same time it is the image of all the elements of which the point P lies in the intersection of o with α , for r is projected out of O by any plane which contains r . The

¹) A fine representation of the line elements of α on the points of space may be found in the thesis of Dr. G. SCHAAKE. (*Afbeeldingen van figuren op de punten eener lineaire ruimte*, P. Noordhoff, 1922).

plane pencil (O, ω) is accordingly the image of the null system $N(0,1)$, in which N lies on b .

Let g_1 be a straight line of H cutting b_1 and a_1 so that its projection \bar{g}_1 passes through B_2 . As any point of g_1 may be considered as a point of contact R , P is an arbitrary point of \bar{g}_1 and g_1 is the image of all line elements lying on \bar{g}_1 . The straight lines of the scrolls (g_1) and (g_2) are therefore *singular tangents*.

3. Let the symbol (λ, π) indicate a system of line elements (P, l) in which the points P lie on a curve of the order π and the straight lines l envelop a curve of the class λ .

The image of a *plane pencil* $(1,0)$ is apparently a *plane pencil* of tangents. If P lies in A , the plane pencil (r) coincides with the plane pencil (A, l) . The plane pencils (B_1, l) and (B_2, l) are represented by congruences $(1,1)$ (cf. § 2).

The image of a *system* $(0,1)$ consists of the tangents of a conic λ^2 lying in the projecting plane of the fixed straight line l .

A *system* $(1,1)$ consists of the line elements of which P lies on a straight line c and l passes through a point D . If P moves on the straight line c , R describes a conic γ^2 (through O) and ϱ envelops the tangent cone which has the pole of the plane γ of γ^2 as vertex. The plane $\sigma \equiv Ol$ revolves round $d \equiv OD$ and describes a pencil which is projective with the system of the tangent planes ϱ (index 2). The image lines r describe accordingly a *cubic scroll* of which d is the double directrix and γ^3 a director curve.

The intersection of this scroll $(r)^3$ and the plane γ consists evidently of the conic γ^3 and the tangent σ which rests on c and is the image of the line element (B, b) belonging to $(1,1)$. The points of intersection of γ^3 with c lie on the straight lines a_1 and a_2 ; the line elements to which they belong, are represented by the tangents of $(r)^3$ which, apart from σ , rest on c . To $(r)^3$ there belong two straight lines of H ; they cut each other on d , and are the images of the line elements for which l passes through B_1 or B_2 .

4. Let a *system* (λ, π) be given. The curve (P) which is of the order π , is projected out of O by a cone of the same order, which cuts H along a curve (R) of the order 2π (with a double point in O). The polar plane of the point E , chosen at random, contains accordingly 2π points R ; hence the tangent planes ϱ envelop a surface of the class 2π . To each plane ϱ there corresponds one plane (Ol) ; inversely to one plane Ol (containing π points P) there

are conjugated π planes ϱ . The planes Ol and the tangent planes ϱ define on any straight line a correspondence with characteristic numbers 2π and 2π . Through each coincidence there passes one image line r ; accordingly the system (λ, π) is represented by a scroll of the order $(\lambda + 2)\pi$.

A system (λ, π) contains 2λ straight lines l passing through B_1 or through B_2 . As each of them carries π line elements, the scroll contains 2λ straight lines of the hyperboloid, each of which is a π -fold straight line of the scroll.

The system $(1, \pi)$ in which the points P form a curve (P) of the order π which has a κ -fold point D and where all straight lines l meet in D , has to be examined separately. For here a plane Ol contains only $(\pi - \kappa)$ points P and defines therefore only $(\pi - \kappa)$ planes ϱ . The characteristic numbers of the correspondence between the points of a straight line are in this case $(\pi - \kappa)$ and 2π , so that the system $(1, \pi)$ is represented by a scroll of the order $(3\pi - \kappa)$ on which the straight line OD is evidently 2π -fold.

A system $(1, \pi)$ of the kind in question is found in a null system $N(\mu, \nu)$ which is the locus of the null points of the rays of a plane pencil round a point D . For this null curve is a curve of the order $(\mu + \nu)$ with a μ -fold point D , so that the line elements form a system $(1, \mu + \nu)$.

5. A null system $N(\mu, \nu)$ is represented by a congruence of rays $[r]$. The straight line a_1 is a null ray for ν of its points P and the straight line r representing (P, a_1) , coincides with a_1 . Hence a_1 and a_2 are ν -fold rays of the congruence; the field-degree of $[r]$ is accordingly 2ν .

Let Q be the central projection of the point F . The null curve of Q is projected by a cone of the order $(\mu + \nu)$ and this cone has $2(\mu + \nu)$ points R in common with the conic which is the intersection of H and the polar plane of F . From this follows that the sheaf-degree of the congruence is $2(\mu + \nu)$. The image of an $N(\mu, \nu)$ is therefore a congruence $(2\mu + 2\nu, 2\nu)$.

Accordingly a bilinear null system $N(1, 1)$ is represented by a congruence $(4, 2)$. The singular points S_1, S_2, S_3 define three points R_1, R_2, R_3 on H ; these are the vertices of three plane pencils $(r_1), (r_2), (r_3)$, representing the plane pencils round the points S , hence singular points of the congruence $[r]$. The line elements on the three singular straight lines $s_1 \equiv S_1S_2, s_2$ and s_3 are represented by the tangents of three conics σ_k^2 through O . Their planes σ_k are singular planes of the congruence. Also the plane $\sigma \equiv R_1R_2R_3$ is singular

for it contains one ray of each of the plane pencils (r_k). All tangents of the conic σ^2 along which H is intersected by σ , belong therefore to $[r]$. On σ^2 there lies one point B_1^* of b_1 and one point B_2^* of b_2 . These two points are also *singular*, for the tangent to σ^2 at B_1^* is the image of the line element of $N(1,1)$ that has its null point in B_1 ; but this line element is represented by *any* ray of the plane pencil (r) round B_1^* .

The null point of the straight line b is represented by the plane pencil (O, ω); hence also O is a *singular point* of the congruence (4,2).

6. The *enveloping cone* with vertex F is the image of a system of ∞^1 line elements of which the points P lie on the conic π^2 , which is the central projection of the conic q^2 in the polar plane of F . The straight lines l pass through the projection Q of F . Any line l is the projection of a conic through O and contains therefore two points P , corresponding to the two points R of q^2 in Ol . The cone round F has accordingly a system (1,2) for image. The conic π^2 passes through B_1 and B_2 , the point Q is to be counted double, being the class curve of l .

If F describes the straight line f , the corresponding tangent cones form a *congruence* (2,2) with *directrix* f . The curves of contact q^1 pass through the intersections S_1^*, S_2^* of H with the polar line of f , and rest on b_1 and b_2 . Hence the curves π^2 form a pencil with the base points B_1, B_2, S_1, S_2 , which are *singular null points*. Through a point P there passes one line l ; for the corresponding point R carries one tangent r that rests on f and has the straight line $l \equiv PQ$ for projection.

A straight line l defines a point Q of the projection q of f , hence a point F , and through this there pass two tangents r to the conic in Ol . The *congruence* in question (2,2) is therefore represented by a *null system* $N(1,2)$.

The line f cuts the tangent plane $\omega \equiv b_1 b_2$ in a point F^* , the projection S of which lies on b and is a *singular null point* because the tangent OS represents all line elements round S .

The intersections F_1^* and F_2^* of f and H are singular for the congruence (2,2); their projections F_1 and F_2 on α are therefore *singular null points*.

In this way the *seven singular null points* which $N(1,2)$ must have¹⁾, are indicated.

¹⁾ Cf. e.g. my paper on plane linear null systems. These Proceedings Vol. XV p. 1165.

Through F_1^* there pass two straight lines g_1 and h_1 of H , through F_2^* two straight lines g_2 and h_2 . These four lines form a skew quadrilateral; g_1 and g_2 cut each other in S_1^* , h_1 and h_2 in S_2^* ; g_1 and h_1 rest on b_1 , g_2 and h_2 on b_2 . The projections \bar{g}_1 , \bar{h}_1 , \bar{g}_2 , \bar{h}_2 of these lines are evidently *singular null rays* and form a quadrilateral which has the singular null points S_1, S_2 ; F_1, F_2 ; B_1, B_2 as angular points. For $B_1 \equiv \bar{g}_2 \bar{h}_1$, $B_2 \equiv \bar{g}_1 \bar{h}_2$; $F_1 \equiv \bar{g}_1 \bar{h}_2$, $F_2 \equiv \bar{g}_2 \bar{h}_1$; $S_1 \equiv \bar{g}_1 \bar{g}_2$, $S_2 \equiv \bar{h}_1 \bar{h}_2$.

The plane O_f cuts H along a conic, the tangents of which belong to $[r]$; hence the straight line q , (the projection of f) is a *singular null ray*. On q lie the singular null points F_1, F_2 and S . But S is the intersection of a tangent o , therefore also a point of the *singular null ray* $b \equiv B_1 B_2$. Accordingly the singular elements of $N(1,2)$ form the figure of the angular points, the diagonal points and the sides of a *complete quadrangle*. This null system is therefore of the same kind as the $N(1,2)$ which arises if to each straight line there are conjugated as null points its intersections with the conic in which it is transformed by an involutory quadratic correspondence.¹⁾

7. Five tangents r define a *linear complex* A ; this has a congruence (2,2) in common with the complex of the tangents of H . The representation on α is again a *null system* $N(1,2)$; for a point P defines a point R and in ϱ there lies one ray of the plane pencil which in A has the null point of ϱ as vertex; and a line l defines on H a conic of which two tangents belong to the linear complex.

This complex has two straight lines in common with each of the scrolls of H ; they form a skew quadrilateral g_1, g_2, h_1, h_2 , the angular points of which are *singular points* for the congruence (2,2). For in A the point g_1, g_2 is the null point of the plane ϱ defined by g_1 and g_2 , so that any tangent at that point belongs to both complexes. Consequently the points $\bar{g}_1 \bar{g}_2$, $\bar{g}_2 \bar{h}_1$, $\bar{h}_1 \bar{h}_2$, and $\bar{h}_2 \bar{g}_1$ are *singular null points* of the null system (1,2) in α .

As g_1 and h_1 rest on b_1 , \bar{g}_1 and \bar{h}_1 pass through B_2 ; hence B_1 and B_2 are *singular null points*. Also here the six null points are the angular points of a complete quadrilateral the sides of which are *singular null rays*. The plane pencil (O, ω) contains one ray of A which therefore also belongs to the congruence; its intersection S is the *seventh singular point* of $N(1,2)$. As S lies on b and B_1 and B_2 are singular, also b is a *singular null ray*.

¹⁾ The general null system (1,2) has no singular null rays (l.c. p. 1167).

8. With a complex of the n^{th} order, I^n , the complex $\{r\}$ of the tangents has a *congruence* $(2n, 2n)$ in common which has for image a *null system* $N(n, 2n)$. I^n has $2n$ straight lines in common with any scroll of H ; hence the null system has $4n$ *singular straight lines*, $2n$ of which pass through B_1 and $2n$ through B_2 . B_1 and B_2 are therefore *singular null points*. The straight line b is evidently a *singular null ray*.

Anatomy. — "*The ontogenetic development of the Corpus striatum in birds and a comparison with mammals and man*". By Dr. C. U. ARIËNS KAPPERS.

(Communicated at the meeting of November 25, 1922).

In the last ten years the corpus striatum has been a centre of interest as well for anatomists as pathologists, the latter chiefly after the researches of KINNIER WILSON.

There are however great differences in the intraventricular growths to which this name is given in different vertebrates.

Though I shall deal here chiefly with the corpus striatum in birds, mammals and man, I will start with making some introductory remarks on the intraventricular growths in fishes since the same principle which we shall meet in the amniota is already observed here: viz. the fact that the so called striatal parts do not only arise from the base of the forebrain but also from the mantle.

If one looks at the forebrain of a teleost or ganoid, it seems as if only the basal part of the forebrain consisted of nervous tissue, whereas the dorsal part merely consists of a choroid membrane.

This however is only seemingly so.

As a matter of fact, the two primordia generally observed in forebrains, the basal one and the dorsal one (from the latter of which the mantle arises), are both present also in embryos of Teleosts and Ganoïds.

Whereas however the dorsal part in other fishes enlarges in a mantle-like way, increasing chiefly in surface and folding inward, the mantle primordium in Teleosts develops in a quite different way. Instead of increasing in surface it increases in thickness, thus narrowing the ventricle of the forebrain in which it protrudes.

This increase in thickness even goes so far that the pallial part bulges outward, pushing the dorsal wall latero-ventrally, in consequence of which the roof membrane is stretched and widely extended from left to right.

Thus an everted pallium is formed in these fishes, in contrary to the inverted mantle of other animals.

This process of development is seen in all larvae of Teleosts, and clearly demonstrated by a study of *Lepidosteus osseus* (a bony ganoid)

of which I give here some pictures. In the first figure (*Lepidosteus* larva of 5 c.M.), the limit between the basal primordium (from which the palaeostriatum arises), and the dorsal (pallial) primordium is indicated by a line, the dorsal point of which might even be drawn somewhat more laterally (to coincide with the fiss. endorhinalis interna). The basal *point de repère* of this line lies in the fissura endorhinalis externa, only slightly indicated in this stage.



Fig. 1. Transverse section of the forebrain of a 5 c.M. larva of *Lepidosteus*.

N.b. = basal nucleus or nucleus peduncularis anterior.

The pallial part is very small in this stage. In a later stage, the pallial part however increases considerably. In fig. 2 and 3 I have given transsections of a 10 c.M. larva and a full grown animal (1.20 M long). These two latter figures represent a more frontal level than figure 1, so that the olfactory bulb is cut, in order to show the reader that here we have really to do with a pallial part, (p.), which however in these fishes does not grow like a real mantle, but merely increases in thickness. The insertion of the roof membrane is at the place of the X in fig. 3, from which results that nearly all the mantle substance has an intraventricular position.

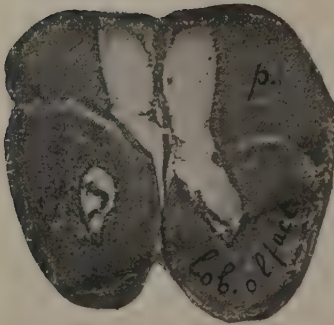


Fig. 2. Transverse section of the forebrain of a larva of *Lepidosteus* (10 c.M.).

p = pallium.

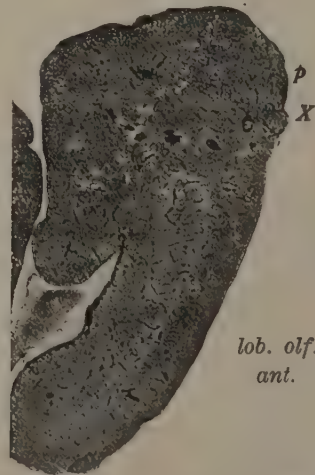


Fig. 3. Transverse section of the forebrain of an adult *Lepidosteus* (right half) *x* = insertion of the roof membrane, *p* = nervous pallium.

This increase in thickness gives rise — a little more caudally than fig. 3 — to a large mass of nervous tissue, extending over the palaeostriatum (which itself is derived from the basal part) and therefore has been named by EDINGER *epistriatum*.

EDINGER himself thought that this *epistriatum* is an outgrowth of the striatum. I have however been able to show that it really is caused by a medial thickening of the pallium extending over the palaeo-striatum. Also by studying its fibre-connections — which appear to be homologous to the fibre connections of the selachian mantle — I have been able to show this homology.¹⁾ Referring for further details concerning the Teleostean brain to the works of JOHNSTON²⁾, SHELDON³⁾, VAN DER HORST⁴⁾ and HOLMGREN⁵⁾, I will only call the attention to the fact that this *epistriatum* of fishes has chiefly primary olfactory functions, viz. that it receives chiefly fibres of the tr. olfactorius (fibrae bulbo-epistriaticae). In this sense it is a *primary epistriatum*.

A *primary epistriatum* also develops in Amphibia but it remains very small there (receiving only tr. olfact. fibres from the bulbus accessorius⁶⁾) since the surface growth of the mantle is so considerable in Amphibia. This primary *epistriatum* of Amphibia develops entirely independently from the palaeo-striatum or basal nucleus, in front of it, from the side wall of the forebrain.

In Reptilia the primary *epistriatum* is superposed by a much larger *secondary epistriatum* or *archistriatum* i. e. by an ingrowth of the mantle which does not receive bulbo-epistriatic fibres but lobo-epistriatic, i. e. secondary olfactory fibres from the primary olfactory cortex (palaeocortex mihi; cortex praepiriformis BRODMANN).

Notwithstanding its enormous development and intraventricular

¹⁾ The structure of the Teleostean and Selachian brain. Journ. of Comp. Neur. Vol. XVI, 1906. Zur vergleichenden Anatomie des Vorderhirns der Vertebraten, Anat. Anzeiger Bnd. XXX, 1907.

²⁾ The telencephalon of Ganoids and Teleosts. Journ. of Comp. Neur. Vol. XXI, 1911 and the Teleostean Forebrain, Anat. Record. 1912.

³⁾ The olfactory tracts in Teleosts. Journ. of Comp. Neurology Vol. XXII, 1912.

⁴⁾ The forebrain of the Symbranchidae. Proceedings of the Kon. Akademie v. Wetensch. Amsterdam, 1920.

⁵⁾ Zur Anatomie und Histologie des Vorderhirns und Zwischenhirns der Knochenfische, Acta Zoologica, Bnd. I, 1920.

⁶⁾ HERRICK. The morphology of the forebrain in Amphibia and Reptilia. Journ. of Comp. Neurol. Vol. XX, 1920.

DE LANGE. Das Vorderhirn der Reptilien, Fol. Neurob. Bnd. V, 1911.

ARIËNS KAPPERS and HAMMER. Das Zentral-Nervensystem des Ochsenfrosches (*Rana Catesbyana*) Psych. en Neur. Bladen 1918.

(hypopallial, ELL. SMITH¹⁾ growth, extending far backward, where it is continuous with the piriform and ammonocortex, this archistriatum keeps its contact with the olfactory area in front of the Foramen Monroi, near the primary "Anlage" of the epistriatum (nucl. tr. olfact. lateralis in Reptilia: CROSBY²⁾).

One might be inclined to ask, how it is possible to ascribe this hypopallial growth to neurobiotaxis — as ELL. SMITH does — if the majority of aferent fibres (tr. cortico-epistriaticus) comes from the periphery

Such fibres indeed cannot account for this mode of growth. But the archistriata (sec. epistr.) of both sides are connected by a very strong commissure, which thus provides them with medial impulsus and moreover it receives aferent fibres from the basimedial grey by the taenia terminalis fibres. Both systems must be made responsible for the medial intraventricular growth of the archistriatum.

Whilst this archistriatum which is thus derived from the innerside of the mantle (*hypopallium* ELL. SMITH³⁾) forms the larger part of the intraventricular mass in Chelonia (where the paleoatriatum is but small) a new striate substance which is only very small in turtles, becomes evident in Lacertilia, Ophidia and Crocodilia: the *neostriatum*. Moreover the paleoatriatum, the original basal nucleus of the forebrain, enlarges considerably in these animals (*palaeostriatum augmentatum* or *mesostriatum*).

Whereas the *palaeostriatum augmentatum* is really an increase from the same matrix from which the paleoatriatum primitivum arises, and from its immediate surrounding (corresponding approximately with the tuberculum parolfactorium) the neostriatum is an entirely new addition starting in Reptilia as I pointed out in 1908⁴⁾. It arises from two sources. 1°. from the base of the brain in front of the paleoatriatum and 2° from the latero-frontal mantle joining this region, as has been pointed out by ELL. SMITH (l. c.). The paleoatriatum, but chiefly the neostriatum receives its stimuli from the tweenbrain and this may be the neurobiotactic cause of its intraventricular medio-caudally directed growth.

The neostriatum together with the archistriatum (which is separated from it in Ophidia and Lacertilia by a deep fissure, the fiss. strio-archistriatica), is called *hypopallium* by ELL. SMITH, on account of their character as an ingrowth of the pallium.

¹⁾ Vida infra.

²⁾ The forebrain of Alligator mississippiensis, Journ. of Comp. Neur. Vol. 27, 1917.

³⁾ A preliminary note upon the morphology of the corpus striatum. Journ. of Anat. (English), Vol. LIII, 1919.

⁴⁾ Die Phylogenese des Rhinencephalons, des Corpus Striatum und der Vorderhirn-commissuren. Folia Neurobiologica Bnd. I, 1908.

Weitere Mitteilung zur Phylogenese des Vorderhirnes und des Thalamus, Anat. Anzeiger Bnd. 1908.

There is no doubt indeed that the neostriatum partly arises as such a hypopalial ingrowth in all the higher vertebrates, though its anlage is not limited to the mantle, but, also extends over the base of the brain in front of the palaeostriatum (immediately behind the anterior olfactory ventricle).

Whilst the neo-striatum is separated from the archi-striatum by the fissura *strio-archistriatica* (see my book on the Comp. Anatomy of the brain, Vol. II fig. 534), ELLIOT SMITH has rightly pointed out that the boundary between the neo-striatum and palaeostriatum is chiefly indicated by blood vessels. I may add that besides a shallow groove may indicate this boundary line (also in Reptilia), which groove I shall call *fissura neo-palaeostriatica*.

I have now studied the ontogenetic development of the different parts of the striatum complex in birds, mammals and man, and shall give here a short review of it, leaving the archi-striatum further out of discussion, since its place in brain-anatomy as the homologue of the nucleus amygdalae of mammals is since long established.

Starting than with birds I may remind that practically all anatomists have accepted the division of the forebrain of these animals as given by EDINGER in 1896.

Underneath the pallium (in which the cortex is very primitive) and continuous with it, is the *hyperstriatum*, forming the most dorsal and most lateral part of the striate complex. This hyperstriatum is in most birds — not in all —, easily distinguished in two divisions, by a thin medullary lamella: the *lamina medullaris hyperstriati*. These divisions I shall call *hyperstriatum superius*¹⁾, and *hyperstriatum inferius*²⁾.

The hyperstriatum inferius in its lateral part shows a special field characterized by large cells, and richly provided with medullary fibres: the *ecto-striatum* of authors, which like the rest of the hyperstriatum is separated from the underlying *meso-striatum* (palaeostriatum augmentatum) by the *lamina medullaris dorsalis* of authors, which I prefer to call *lamina medullaris externa* since it does not only form the dorsal but also the lateral boundary of the meso-striatum. This lamina medullaris externa is very richly provided with bloodvessels as is also observed by HUNTER (Sydney) in the Kiwi.

¹⁾ This was called by SCHROEDER pars fronto-dorsalis hyperstriati. It consists of the areae A. C. and D. of Rose's (c.f. SCHROEDER: Der Faserverlauf in Vorderhirn des Hühnes, Journ. of Psych. und Neur. Bnd. 18, Erg. Heft 1912, and Rose „Die zytolectonische Gliederung des Vorderhirns der Vögel". Ibidem Bnd. 21, 1914),

²⁾ This corresponds with the areae G¹, G², G³ of Rose's and with the striatum parichale of Kalisher (Comp. Kalisher: Abhandl. der Akad. der Wissensch. Berlin. 1900, 1901, 1905).

Besides the boundary of the mesostriatum and hyperstriatum in some birds is marked on the ventricular side by a slight groove, my *fissura neo-palaeostriatica*.

In caudal direction the mesostriatum, which extends to the ventricular surface becomes smaller and smaller, thus exhibiting a sort of cauda, which follows for some distance the caudal pole of the hyperstriatum inferius, and may be called *substantia palaeostriatica caudata* (see fig. 11 and 12).

In some large birds, like Pelicanus, the hyperstriatum and meso-striatum may be separated from each other — starting at the ventricular side — by an obtuse object without cutting, which probably is due to the medullary external lamella being so richly provided with bloodvessels.

In the centre of the mesostriatum (or palaeostriatum augmentatum) the so called *basal nucleus* of authors (palaeostriatum primitivum) is found, a cluster of large cells, separated in front of the augmented part of the palaeostriatum by another lamella the lamina medullaris ventralis of authors, *lamina medullaris interna mihi*.

The archistriatum or nucleus amygdalae of which I shall not speak here further is pushed backward and ventrally in birds by the enormous development of the hyperstriatum. Consequently the fissura strio-archistriatica, so conspicuous in Lacertilia and Ophidia, has become invisible in birds (as is already the case with Crocodiles).

In order to study the embryonic development of these parts in birds, I made use of haematoxyline and silverseries of the chick of 4, 5, 5½, 6, 7, 9 and 11 days of incubation and of an embryo of the ostrich some days before birth.

In a five days embryo of a chick, we find in a transverse section made on the level of the foramen Monroi, four protrusions in the ventricle (fig 4). The lower protrusion *a* is the *eminentia basimedialis* which some sections more frontally continues in the septum. This forms the basi-medial grey substance and has not to do with the striate complex.

The other three protrusions form parts of the so called striate complex.

The protrusion *b* is the primordium of the *palaeostriatum*. Its centre (less dark in fig. 4), is the basal nucleus or *palaeostriatum primitivum*, which is augmented by the surrounding darker cells, the *palaeostriatum augmentatum*.

This protrusion has only a small frontal extension (as is seen in the sagittal section, represented in fig. 5. It is chiefly confined to the level of the foramen Monroi and continues backward in the side wall of the recessus praeopticus (r. o. fig. 5). The protrusion *b* is separated by a fissure (the *fissura neo-palaeo-striatica*) from the

tuberculum *c* which is less protruding but continues further frontally than *b*, bending down more or less to the base of the brain.

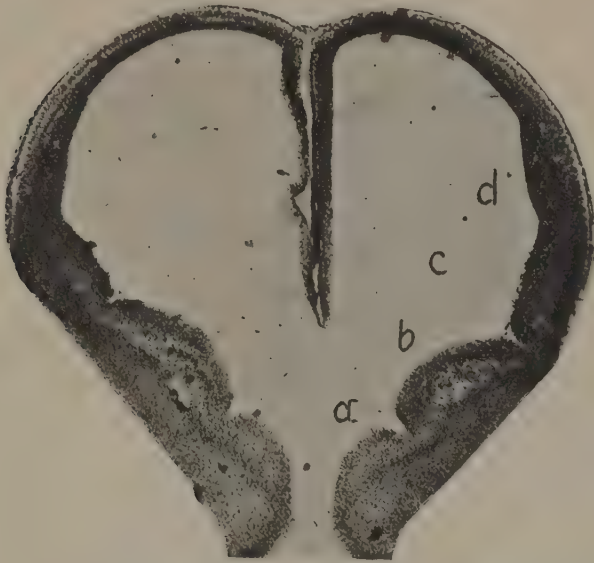


Fig. 4. Transverse section of the forebrain of a chickembryo of 5 days on the level of the foramen Monroi. *b* = primordium of the palaeostriatum, *c* = primordium of the hyperstriatum inferius, *d* = primordium of the hyperstriatum superius. For *a* see text.

This protrusion *c* appears to be the primordium of the *hyperstriatum inferius*. Caudally the groove which separates it from *b* fades away, the cells of *c* extending over *b* (comp. also fig. 6).

Dorsally from *c*, arising equally from the mantle is *d*, merely a thickening of the pallium in this stage which however appears to give rise to the *hyperstriatum superius*.

Figure 6, representing a sagittal section, is taken from an embryo of six days of incubation. The section shows the relation of the *hyperstriatum inferius* *c* to the palaeostriatum augmentatum *b*, which extends frontally to the triangular fissure, a part of the fissura *neo-palaeostriatica*.

It is further seen that *c*, the hyperstriatum inferius, arises on this level from the basal region in front of the palaeostriatum corresponding with the tuberculum olfactorium (t. o.). The hyperstriatum inferius thus partly has a basal origin (*partly* because more laterally it is continuous also with the mantle as we already saw in the transverse section of fig. 4).

Examining the same series on a more lateral level (fig. 7), we meet with the hyperstriatum superius *d*, and see that this arises

from the mantle only, i. e. from the brainwall above the small split that indicates the communication between the lateral ventricle and the olfactory ventricle (already in fig. 6 the frontal part of the pallium shows a thickening at this place).

In the last section of this series which I reproduce (fig. 8), all the parts of the striatum complex of birds are already visible in their mutual arrangement: the hyperstriatum superius (*d*) forming the most dorsal part

and extending over the rest, being continuous frontally with the pallium. Underneath it we find the hyperstriatum inferius *c* being in this

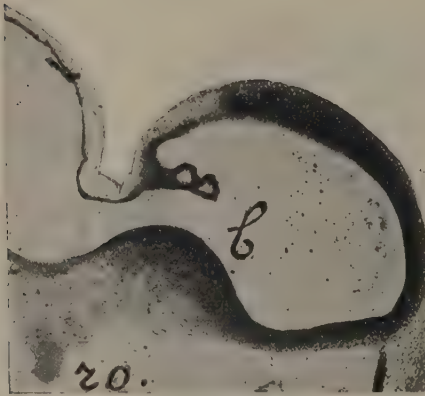


Fig. 5. Sagittal section of the forebrain of a chickembryo of $5\frac{1}{2}$ days.

b = primordium of the palaeostriatum.

r.o. = wall of the recessus opticus.

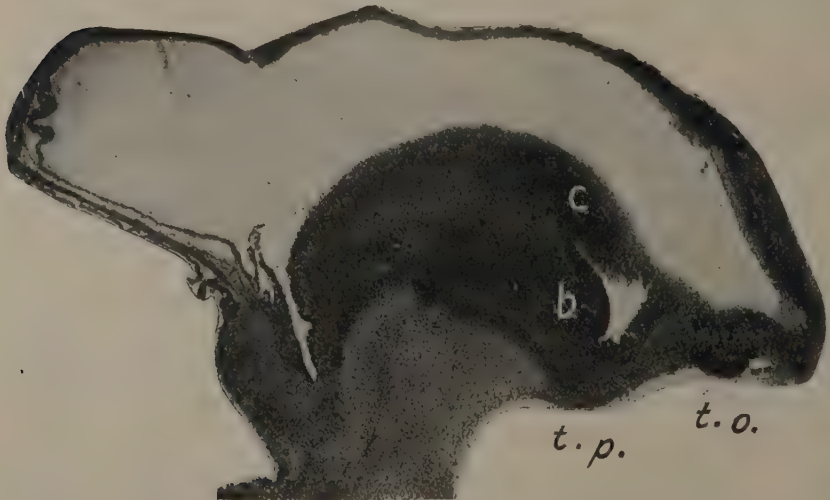


Fig. 6. Sagittal section of the forebrain of a chickembryo of 6 days on a level lateral to fig. 5.

t.p. = tuberculum parolfactorium, *t.o.* = tuberc. olfact.

b = palaeostriatum augmentum, *c* = hyperstriatum inferius. (= mesostriatum).

section continuous with the most frontal part of the basis cerebri (more laterally with the mantle) and covering *b*, the meso-striatum or palaeostriatum augmentatum, in which the lighter centre (richly provided with fibres) is the primitive palaeostriatum, the basal nucleus.

If we now look at the figures of a 11 days embryo of the chick, we find that the chief alteration exhibited, is the enlargement of both parts of the hyperstriatum, which not only have increased in

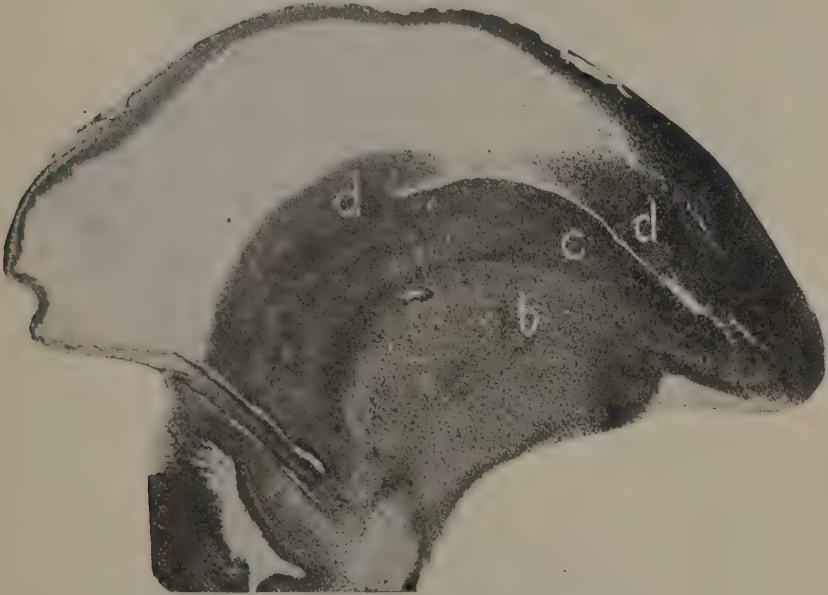


Fig. 7. Sagittal section of the forebrain of a chickembryo of 6 days.

b = palaeostriatum augmentatum (= mesostriatum).

c = hyperstriatum inferius.

d = hyperstriatum superius.

thickness (as appears from the fact that much less of the ventricle has remained free), but also has enlarged in medial direction.

The latter fact is evident from a comparison of figg. 9 and 6, which are taken on approximately corresponding levels (rather medial).

Whereas in fig. 6 on this level nothing is as yet visible of the hyperstriatum, the latter is very clearly shown in fig. 9, as a result of its growth in medial direction, further extending into the ventricle. It also shows the division in hyperstriatum superius and inferius.

In this figure we see moreover that the hyperstriatum superius is continuous only with the brainwall above the ventriculus,

being entirely derived from the mantle¹⁾, not from the basal part of the brain.

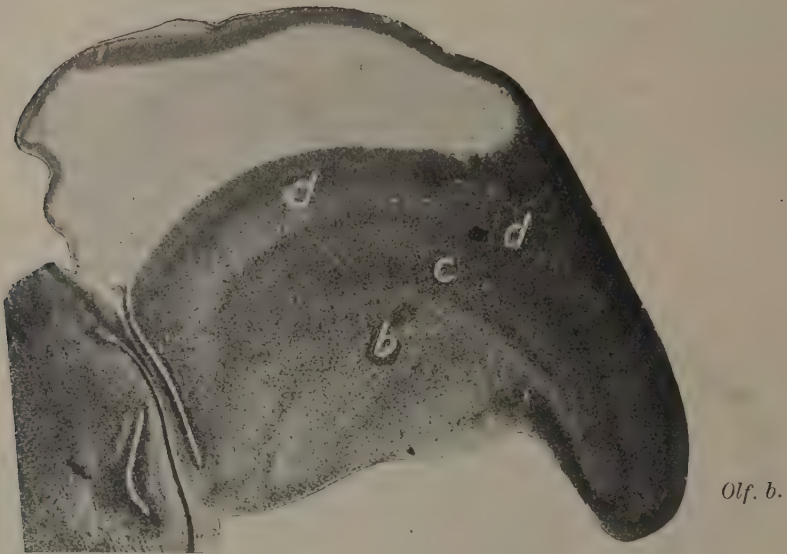


Fig. 8. Sagittal section of the forebrain of a chickembryo of 6 days (lateral to fig. 7) *d* = palaeostriatum augmentatum (= mesostriatum) *c* = hyperstriatum inferius, *d* = hyperstriatum superius.

In fig. 9 only a small part of the palaeostriatum (*b*.) is seen, viz that part which is continuous with the recessus praeopticus.

Fig. 10 is interesting to us because it shows that the hindpole of the striatum nearly only consists of hyperstriatum inferius (*c*), the lamina medullaris hyperstriati (in this stage of development) ending only little beyond the contact of hyperstriatum superius and pallium. In the same figure (but better in 11 and 12) is seen that the hyperstriatum inferius is continuous with the base of the brain (whilst more laterally it is continuous in the pallium).

Of the palaeostriatum besides the part that is continuous with the recessus opticus a frontal part is seen in fig. 10, seemingly separated from the hindpart by a recess of the ventricle. This is however only seemingly so, this aspect being caused by the fact

¹⁾ One might ask if the part called hyperstriatum superius here is not partly the "mediale Sagittal-Wulst" of the cortex with which the hyperstriatum superius in many birds (f. i. the *Cacatua*) coalesces. This however is not so here, though later the hyperstriatum superius continues in the medio-dorsal mantle, without showing any medullary limitation.

that the palaeostriatum following the lateral convexity of the brain is curved and not cut here in its entirety length. In fig. 11 this separation is smaller and in fig. 12 it has entirely disappeared.

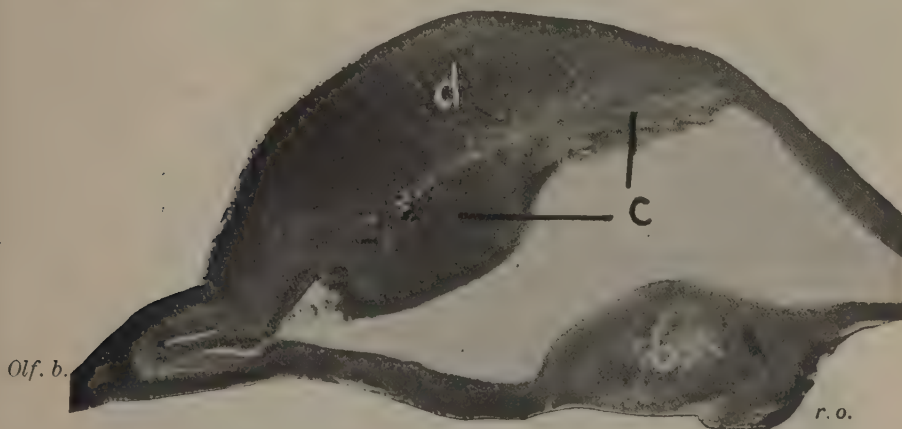


Fig. 9. Sagittal section (rather medial) of the forebrain of a chickembryo of 11 days.

r. o. = transition to the recessus opticus.

b = posterior part of the mesostriatum or palaeostriatum.

c = hyperstriatum inferius.

d = hyperstriatum superius.

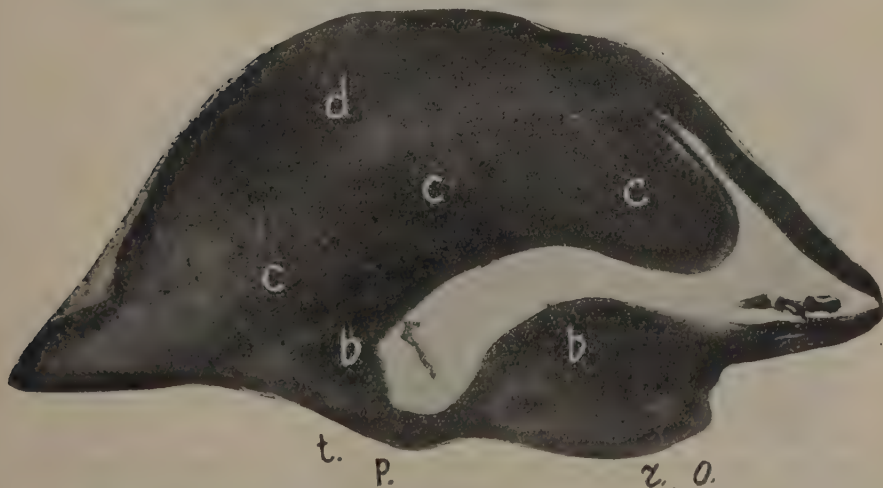


Fig. 10. Sagittal section of an 11 days chickembryo lateral to fig. 9.

d = hyperstriatum superius; *c* = hyperstriatum inferius;

b = palaeostriatum augmentatum (= mesostriatum);

t. p. = tuberculum parolfactorium; *r. o.* = recessus opticus.

The following three figures of this embryo (fig. 11, 12 and 13), show very clearly the presence of the lamina medullaris externa between the hyperstriatum inferius *c* and the *palaeostriatum b*, and also the fact that this lamina is a place of predilection for blood-vessels (*v. s.* = vasa sanguinea). In fig. 11 this lamina has become specially clear by the retraction of the tissue (these are silverpreparations), which retraction finds a natural place of predilection at this spot on account of the loose character of this lamina to which I already referred.

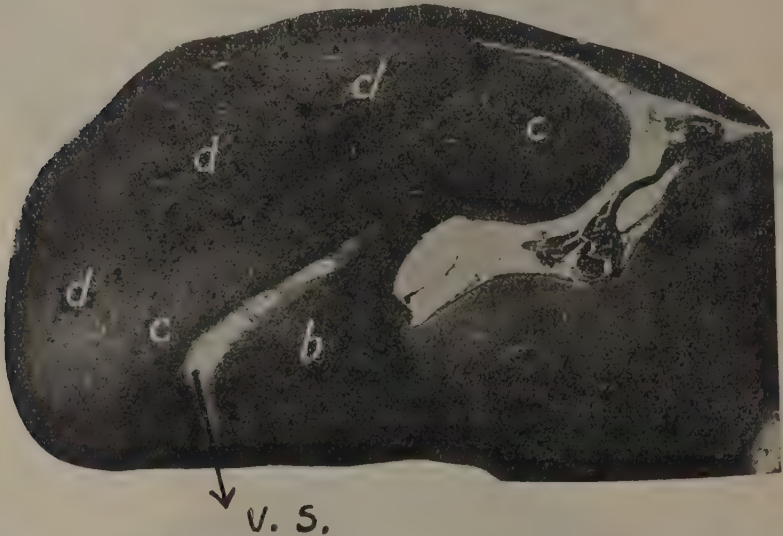


Fig. 11. Sagittal section of a chickembryo of 11 days, lateral to fig. 10.

By the retraction of the tissue the vascular cavities (*v. s.*) in the lamina medullaris externa are very evident.

d = hyperstriatum superius, *c* = hyperstriatum inferius,
b = palaeostriatum augmentatum = mesostriatum).

Figures 12 and 13 moreover show us that the hyperstriatum superius *d* diminishes in lateral direction while *c* enlarges acquiring its connection with the frontal pallium, near the ectostriatum (E. S.).

It is further of interest to note in fig. 11 and 12 that a part of the palaeostriatum *b* continues with and underneath the hyperstriatum inferius bending backward over the recess of the ventricle (above the secondary epistriatum or archistriatum E.).

The caudal enlargement of the palaeostriatum with and underneath the hyperstriatum is what I have called the *substantia palaeostriatica caudata*.

In fig. 13 we see the division of the palaeostriatum by the lamina medullaris interna, the inner segment of the palaeostriatum being the basal nucleus or palaeostriatum primitivum. In this figure also

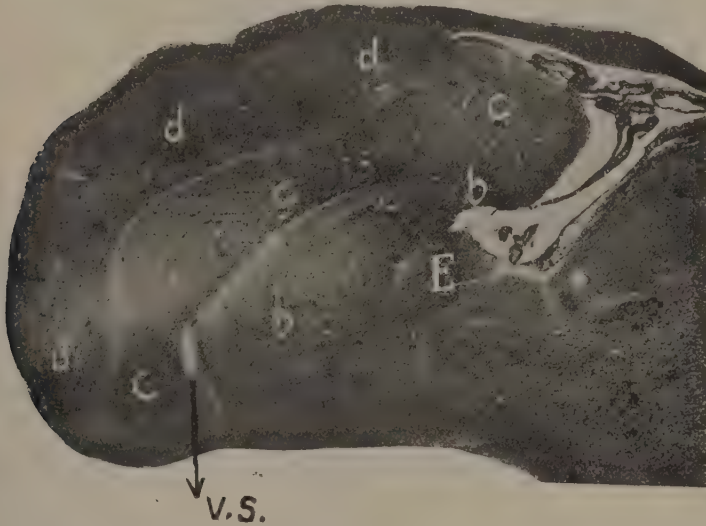


Fig. 12. Sagittal section lateral to fig. 11. Note the dorso-caudal tail of the mesostriatum (*b*) the substantia palaeostriatica caudata, underneath the caudal pole (*c*) of the neostriatum.

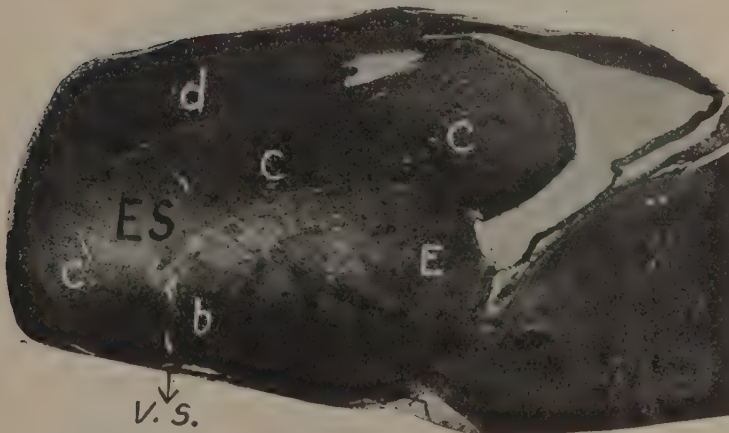


Fig. 13. Sagittal section of the forebrain of an 11 days chick embryo lateral to fig. 12. The hyperstriatum superius (*d*) is smaller, the hyperstriatum inferius (*c*) larger here than in fig. 12. The latter shows its transition in the pallium. *ES* = ectostriatum, *E* = second epistriatum or archistriatum; between the latter and *b* the basal nucleus.

the groove between hyperstriatum *c* and archistriatum E (the fissura strio-archistriatica) is visible (but not indicated).

Resuming my results concerning birds, I may conclude that here (apart from the archistriatum or amygdala) at least two chief divisions of the striatum may be distinguished: the *palaeostriatum*, which is enlarged to a *palaeostriatum augmentatum* (or meso-striatum) and which arises entirely from the base of the brain in front of the recessus praeropticus, and the *hyperstriatum* of which the upper part arises entirely from the mantle (*hyperstriatum superius*), while the underpart (*hyperstriatum inferius*), arises from the mantle (laterally) as well as from the base of the brain in front of the palaeostriatum. Both parts of the hyperstriatum thus show the fact, that intraventricular protrusions of striatal type may originate from the pallium as well as from the base of the brain, as I already pointed out for the primary epistriatum in bony fishes, and as was pointed out by ELL. SMITH for the neostriatum of Reptiles.

Before dealing with the question whether the hyperstriatum superius of birds is included in the neostriatum of mammals (as the hyperstriatum inferius is), or if it is homologous to the claustrum, I shall shortly describe the embryonic development of the striate body in the rabbit and in man, about which already HIS¹⁾, HOCHSTETTER²⁾ and Miss HINES³⁾ have given us such valuable informations.

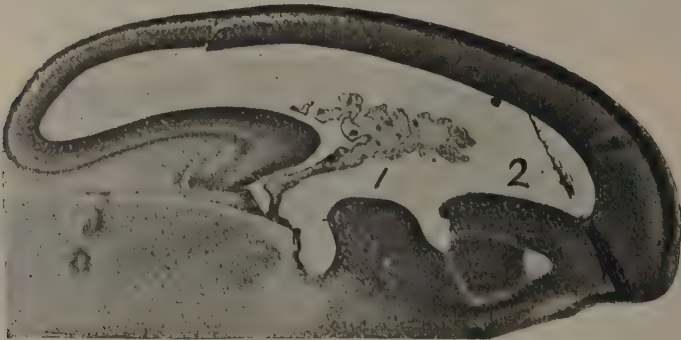


Fig. 14. Sagittal section of the forebrain of a rabbit-embryo of $2\frac{1}{2}$ c.M. total length.

1 = contains ventrally the palaeostriatum, arising on the level of the foramen Monroi.

2 = the neo-striatum arising partially from the mantle.

¹⁾ HIS, Die Entwicklung des menschlichen Gehirns, Leipzig 1904.

²⁾ HOCHSTETTER. Beiträge zur Entwicklung des menschlichen Gehirns. Deuticke, Wien 1920.

³⁾ HINES. Studies in growth and differentiation of the telencephalon in man. Journ. of comp. Neur. Vol. 34, 1922.

In a sagittal section of the brain of a rabbit of $2\frac{1}{2}$ c.M. (fig. 14), we see two proliferation centres of striatum cells. The centre of proliferation marked with 1 contains archistriatic cells covering the primordium of the *palaeostriatum*, the latter being its ventral part arising from the base of the brain about the level of the foramen Monroi, and being continuous with the wall of the preoptic recess. In front of this and arising partly from the base of the

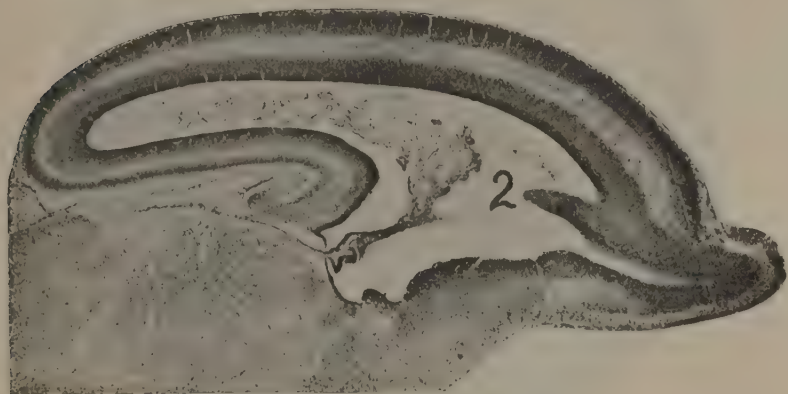


Fig. 15. Sagittal section of the forebrain of a rabbit-embryo of 4 c.M. total length. At 2 the transition of the neostriatum in the deep layer of the frontal-pallium above the olfactory ventricle is seen.

brain, partly from the mantle, we see a part of the anlage of the neostriatum, marked with 2¹⁾.

Examining more lateral sections one sees that 2 enlarges backward and unites with the anlage 1.

I will not deal extensively with the mammalian ontogeny but only reproduce here still another section taken from a rabbit embryo of 4 c.M., in which the continuity of the neostriatum (2) with the pallium above the olfactory ventricle is particularly evident (fig. 15).

Also in the human embryo the two parts of the striate body (I do not speak here about the amygdala) are evident, even more so than in the rabbit.

Fig. 16 shows a frontal section through the forebrain of a human embryo of 27 mM total length in front of the Foramen Monroi. At the left side of the figure the two primordia of the striatum may

¹⁾ The cluster of cells between 2 and the base of the brain continue medially into the septum.

be seen, which have been distinguished by His as the *crus epirhinalicum* and the *crus mesorhinalicum*¹⁾.

Both crura are separated by a fissure which until now has been named *fissura intercruralis*, but which may be called *fiss. neopalaeostriatica* since my researches have convinced me that the mesial crus is the primordium of the palaeostriatum whereas the lateral crus is the primordium of the neostriatum.



Fig. 16. Transverse section of the forebrain of a human foetus of 27 m.M. total length. This section being slightly oblique, the right side shows a more frontal level than the left one.

The mesial crus does not extend as far frontally as the lateral one, as the figure — on the right side — shows, where the mesial crus or palaeostriatum²⁾ has already disappeared, the neostriatum

¹⁾ His called the caudo-medial edge of the latter *crus metarhinalicum*, but it is better not to distinguish this as a separate part since it is merely the caudo-medial side of the mesorhinal crus. It is better to speak only of a lateral and medial primordia as also HOCHSTETTER and MISS HINES do.

²⁾ I may mention here that in this embryo of 27 mM: the transitory cavities of the corpus striatum, which ESSICK first described (Carnegie embryologic public. No. 222), as being constant in human embryos from 15—20 mM, and less constant up to 24 mM, were still present. They are confined in my material to the palaeostriatum. I quite agree with ESSICK that they may be due to insufficient drainage of the brain in that stage in which the production of metabolic solutions may surpass the possibility of drainage, the more so since phylogenetically as well as ontogenetically the dual source of production of liquor (choroid plexusses and ependyma on one hand and intra cerebral vessels on the other) is established

anlage continuing still some distance in front, being continuous not only with the base of the brain, but also with the mantle of the frontal pole, immediately above the olfactory ventricle (this is why His has called it *crus epirhnicum*).

The fissure between the neo- and palaeostriatum becomes less and less deep during further development. In an embryo of 27 centimeters, it has become very shallow, by the prepondering development of the neostriatum, which more and more overlaps the palaeostriatum, as we found it also to be the case in birds with the hyperstriatum inferius.

The fissura neo-palaeostriatica may however still be seen in the full-grown human cerebrum (f.i. about the level of the comm. anterior, fig. 17: F.N.P.S.) forming the ventro-mesial border of the caudate nucleus.



Fig. 17. Transverse section through the corpus striatum of an adult man on the level of the comm. anterior (c. a.).
N.S. = Part of the Neostriatum (nu. caud.).
F.N.P.S. = Fiss. neo-palaeostriatica.
P.S. = Palaeostriatum (covered by the taenia semicircul.)

Underneath this fissure runs the *stria-semicircularis*, which covers here some small vestiges of grey substance lying on the ventricular side of the capsula interna and still belonging to the palaeostriatum,

priory to the resorptive function of both choroid plexusses and Virchow-Robin spaces (Compare also my book on Comp. Anatomy of the N. S. p. 820 and WEED Contributions to Embryolog. publ. by the Carnegie instit. Vol V. 1917).

the main mass of which lies laterally to capsula interna forming the *globus pallidus*.

Small stripes of grey substance are occasionally found between the main lateral mass of the palaeostriatum (the globus pallidus), and its mesial vestiges, chiefly between the fibres of the capsula proper and the anterior (olf.) crus of the comm. anterior.

Also other facts prove the homology of the globus pallidus and those vestiges with the palaeostriatum augmentatum (meso-striatum) of birds. So in some animals (*Hypsiprymnus* f.i.), we occasionally find a continuation of the lamina medullaris externa (the limiting layer between globus pallidus and putamen) in the striatal part mesial to the capsula interna, which lamella also medially may be richly provided with bloodvessels.

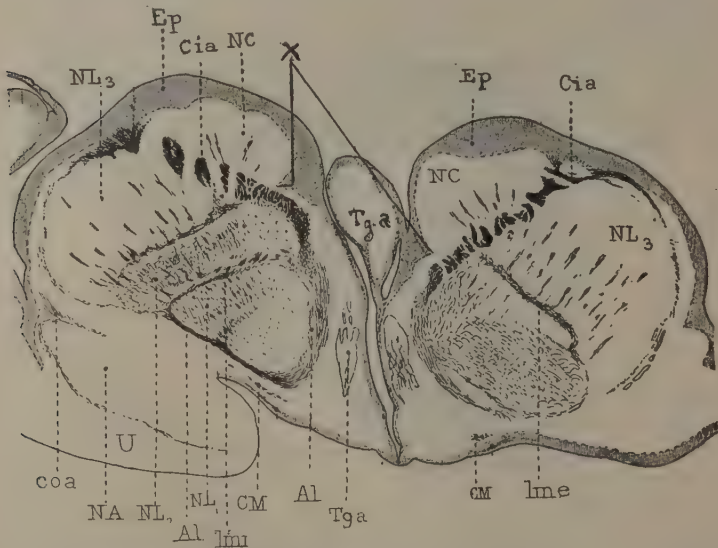


Fig. 18. DEJERINE's case "Longéry". Note the hypertrophy of the ependyma (Ep) specially in the fiss. neo-palaeostriatica at X. — *L. m. e.* = lamina medullaris externa, *L. m. i.* = lamina medullaris interna.

Another method to reinforce this homology is the method used by SPATZ¹⁾, who showed that of all parts of the striatum the globus pallidus obtains the deepest blue if applying sulfur ammonium to the fresh (or formalinefixed) brainmaterial, — on account of its richness in iron.

I have applied this reaction to fresh chicken brains and found

¹⁾ Ueber den Eisennachweis im Gehirn, besonders in Zentren des extrapyramidal motorischen Systems, Ister Teil. Zeitschr. f.d. gesamte Neur. und Psych. Bnd. 77, 1922.

the mesostriatum (palaeostriatum augmentatum) to do the same¹). Whereas however the dark blue colour in the striatum of mammals is confined to the part of the palaeostriatum lying laterally to the internal capsula (the globus pallidus) in birds the deep stain reaches the ventricular side of the palaeostriatum. This difference is apparently due to the accumulation of myelinated fibres in the capsula interna in mammals, myelinated fibres being insensitive to this reaction. Only in such mammals where the capsula is less dense may the blue colour penetrate in it, as SPATZ found to be the case in Ungulates²).

It may be mentioned here, that as in birds (fig. 11 and 12) also in mammals the mesial part of the palaeostriatum may continue some distance caudad underneath the neostriatum, viz. under the caudate nucleus. This *substantia palaeostriatica caudata* accompanies the stria semicircularis on its lateral side, and in some mammals (*Elephas*) is separated from the nucl. caudatus (neostriatum) by a fissure (the continuation of the fissura neo-palaeostriatica), or a medullary lamella with bloodvessels. This may be also observed sometimes in man.

I still will call attention to the fact that the neo-palaeostriatic fissure, generally best indicated on the level of the commissura anterior, may acquire a much more pronounced character in pathology. An example of this is given by the case LONGÉRY described by DEJERINE in his text book³).

In this case the hypertrophied ependyma (Ep. fig. 18), is especially thick at the limit between neostriatum and palaeostriatum (at X in fig. 18), filling up the neo-palaeostriatic groove.

The striatum in this case is further interesting to us, because it shows such a marked similarity with the striatum in birds, which is due to the reduction of the pallium in this case (a hydrocephalic). As a consequence of this reduction the capsular fibres are much less developed than in normal condition, which attributes to the avian aspect of this corpus striatum.

It needs not be repeated here that the division made in the mammalian neostriatum in a putamen and caudate nucleus is not an intrinsic one. The putamen

¹) In birds the deepest blue is shown by the "Sagittalwulst", specially its frontal part, then comes the meso-striatum, then the caudal part of the hyperstriatum inferius. In the thalamus the nucl. rotundus chiefly acquires this colour.

²) That the palaeostriatum is not confined to the globus pallidus alone is also proved by the figures of HOCHSTETTER's embryologic collection, where the early differentiating region is seen to penetrate into the capsula.

³) DEJERINE. Anatomie des centres nerveux, Tome II, fig. 202.

may enlarge medially in such animals (as *Ornithorhynchus*) where the fibres that may be called capsula interna fibres, take a more medial course than usually. Moreover we know that frontally, where the capsula interna fails, the putamen and caudate nucleus fuse and that this fusion is larger the smaller the frontal extension of the capsula is. (E. DE VRIES)¹. Such a fusion may also occur caudally (f. i. in *Elephas*).

Also the separation of the palaeostriatum and neostriatum by the lamina medullaris externa is very evident in man (as also the lamina medullaris interna).

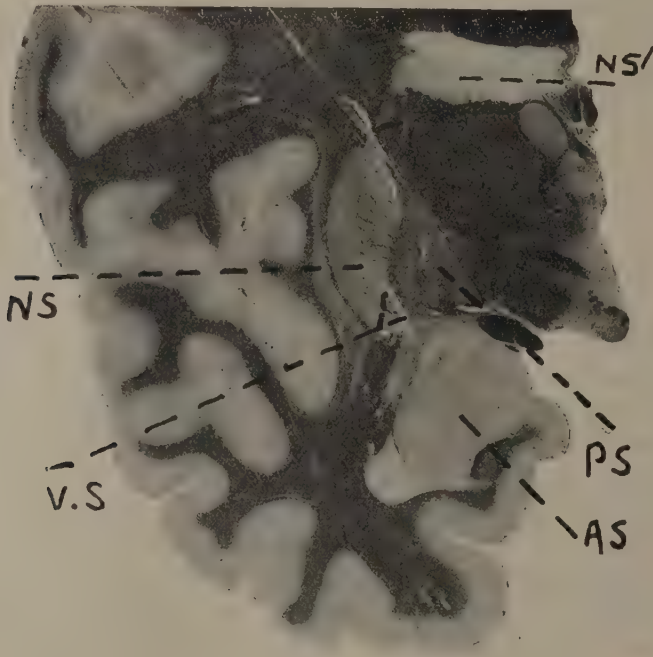


Fig. 19. Neostriatum (*NS* and *NS'*), archistriatum or amygdala (*A.S.*) and palaeostriatum (*P.S.*). *V.S.* = large bloodvessel in the lamina medull. externa. Laterally from the neostriatum (*N.S.*) the claustrum is seen.

At last I want to call the attention to the fact that in the normal human cerebrum the lam. medullaris externa — as in birds — is a place of predilection of bloodvessels more than the other (internal) lamina (vide fig. 19).

Also the connections of the avian striatum (which we know chiefly through the works of BOYCE and WARRINGTON, EDINGER, WALLENBERG, and HOLMES and

¹) E. DE VRIES. Das Corpus Striatum der Säugetiere. Anat. Anzeiger, Bnd. 37, 1910.

SCHROEDER) and those of the mammalian striatum (which are known by the works of MONAKOW, DEJERINE, WILSON, THE VOGTS, RAMSAY HUNT, LEWY), shows in many respects a great resemblance.

In both classes it is chiefly the internal segment of the dorsal thalamus (nucl. anterior and nucl. medialis) together with the ventral thalamus and subthalamic and peduncular region, which are connected with the striate body (as is already the case in Reptiles).

So in mammalia a connection is known to exist between the nucleus anterior thalami and the caudate nucleus (a connection which I can confirm for the marsupials). It seems possible that this connection is homologous or at least analogous to the fibre tract which WALLENBERG showed to exist between the medio-dorsal part of the neo-striatum in birds and the nucl. anterior dorsalis in these animals.

The antero-lateral part of the striatum of birds, which may be homologous to the anterior part of the putamen of mammals has connections which are homologous to the mammalian, if at least the ventral peduncular nuclei of birds are homologous to the corpus subthalamicum and substantia nigra of mammals which is very likely so.

Moreover in both classes this striatal region seems to give fibres to the commissura supraoptica dorsalis of Meynert (which also contains fibres from the palaeo-striatum augmentatum). Concerning the palaeostriatum augmentatum (meso-striatum in birds), it may be that this, partly at least, has to do with trigeminal functions. The considerable enlargement which the original palaeostriatum (small as it is in Amphibia) acquires in Reptilia (chiefly in the Crocodile) and in birds, may be due to projections of the trigeminus, which acquires a very important size and function in the crocodile and is of prepondering importance in birds, the more so since smell and taste are of so little importance here, and the oral sense is so important for life, as I pointed out in 1908. (*Folia Neur.*).

Also the fact that in *Ornithorhynchus* and in *Elephas* (E. DE VRIES), the palaeostriatum is so well developed may support this point of view, since the trigeminus is of prevailing importance here, (the Vth nerve is at least three times larger in Monotremes than in other animals and in the Elephant it provides the trumpets sensibility of muscles and skin). The sensibility provided by the trigeminus to facial muscles is generally of great importance as is proved by the disturbances of tonic innervation of the face, so often seen in man as a consequence of striatal lesions.

As far as concerns birds also ROGER's ¹⁾ experiments seem to prove this conception.

Of course the connections of the striate body are not exhausted with this enumeration, so f.i. there remains to be mentioned the fibre system proceeding from the nucleus ruber to this body as demonstrated by v. MONAKOW and others. It is interesting in this respect that SCHROEDER (l.c.) even mentions — for birds — a direct continuation of the brachium anterius cerebelli, to the palaeostriatum augmentatum of birds. This sort of connections on account of their cerebellar component fall also in the range of motor coordinations. (C.f. also LEWY ²⁾).

The exact character of all these systems has not yet been sufficiently scrutinized,

¹⁾ An experimental study on the corpus striatum in the pigeon. *Journ. of comp. Neur.* Vol. 35., 1922.

²⁾ LEWY, *Die Lehre vom Tonus und der Bewegung.* Jul. Springer, Berlin, 1923.

but so much is true (comp also KINNIER WILSON's¹⁾ experiments and the clinical cases) that the integrating tonetic factor may have an important role in this.

That also visceral disturbances may occur (liver, bladder) in diseases of the corpus striatum and that sympathetic functions have been found to exist here is not so strange in connection with the fact that the primitive striatum develops near the frontal end of the sulcus limitans (which according to several authors, c. f. HERRICK) ends in the preoptic recess. Moreover we know from the researches of BOEKE, DUSSER DE BARENNE, AGDUHR and the BOER that also in muscletonus sympathetic fibres may act a part.

Since visceral and tonetic conditions act an important part in emotions, I would moreover not be astonished if the striatum would prove more and more to have to do with emotions.

At last the question remains if also the *hyperstriatum superius* of birds is included in the neostriatum of mammals and man.

This problem is not easy to solve. One might be inclined to believe that this region of the avian brain is more likely related to the mammalian *claustrum*, a supposition I already made in my textbook.

As the hyperstriatum superius, the claustrum is entirely of pallial origin. Though it may not be derived from (the sixth layer of) the cortex, yet all its cells are derived from a pallial matrix. DE VRIES²⁾ has clearly shown that the claustrum in embryologic stages does not derive from the cortex, but that it derives from the lateral brainwall (which at this spot must be called pallium) between the upperedge of the neostriatum and the cortical layers, separated from the latter by fibres of the capsula extrema.

Also the figures given by LANDAU³⁾ in his anatomy of the fore-brain shows that the way the claustrum develops is that of an intraventricular growth of the pallium (a hypopallial growth in the sense of ELL. SMITH), though apart from the cortex. In its mode of formation the claustrum thus resembles the hyperstriatum superius.

Still in another point there is resemblance between both. Whereas the neo-striatum in mammals (like the hyperstriatum inferius in birds) develops partly from the base of the brain immediately behind the olfactory bulb, partly from the pallium lateral to the olfactory bulb, the hyperstriatum superius like the claustrum only develops from the pallium starting in front immediately above the anterior olfactory lobe of the avian brain.

The fact that the claustrum is very small in Monotremes and Marsupials (where it extends, as in many mammals, partly under-

¹⁾ An experimental research into the anatomy and physiology of the corpus striatum. Brain Vol. 36, 1913 - 1914.

²⁾ Bemerkungen über Ontogenie und vergleichender Anatomie des Claustrums, Folia Neurobiol. Bnd. IV, 1910.

³⁾ Die Anatomie des Grosshirns, Bircher, Bern 1923.

neath the fiss. rhinalis viz. from the region which is covered by the palaeopallium) and only in primates attains a size comparable to birds, does not necessarily contradict this homology, since there are more respects in which the human brain resembles more the avian brain than the lower mammals do, f.i. in its oculo-motor nucleus (comp. Vol. II of my book fig. 261 with figg. 294—295).

Moreover there seems to be a considerable difference in the development of hyperstriatum superius also in birds. As a matter of fact I have not been able to see it in the Kiwi (see HUNTER's work on this peculiar bird ¹⁾ (to be published in the English Journ. of Anat. 1923—24).

In the casnary, and the ostrich it is present, but not yet in the same size as in the chick.

The bloodsupply of the hyperstriatum superius is not in contradiction with such a homology, since the hyperstriatum superius next to many branches of cortical vessels, receives a few branches of the basal arteries, as SHELLSHEAR ²⁾ proved to be also the case with the claustrum of mammals. Perhaps that also the function of these regions shows a certain relation in so far as degeneration of the claustrum seems to give incoordination of movements (athetosis) while also in experimental degeneration of the hyperstriatum superius disturbances of motor function occur (ROGERS). The question may certainly not be considered settled as yet, but the possibility may be kept in mind that the hyperstriatum superius is not involved in the neostriatum of mammals, but in their claustrum, though this region of the telencephalon in birds has taken a development which in this form and size may be peculiar to these animals only just as the large development of the primary epistriatum is peculiar to Teleosts and Ganoïds.

CONCLUSIONS :

Resuming we may state that also the comparative ontogeny of the striate complex in (Reptiles), Birds, Mammals, and Man proves that apart from the secondary epistriatum or archistriatum (amygdala) two chief parts may be distinguished: *palaeostriatum* and the *neostriatum*, which are separated from each other by the *lamina medullaris externa* (richly provided with bloodvessels) and the *fissura-neo-palaeostriatica*.

¹⁾ In connection with this it is interesting to note that in PARKER's figures on the embryology of the Kiwi brain (see Transactions of the Roy. Soc., London) Vol. 182, 183, 1892 and 1893), only two intraventricular primordia (my *b* and *c*, are drawn, and *d* the primordium of the hyperstriatum superius seems to be lacking here).

²⁾ The basal arteries of the forebrain and their functional significance, Journ. of Anatomy (English), Vol. 55, 1920.

The first part contains the basal nucleus or *palaeostriatum primitivum* (its only constituent in Amphibia) and the mesostriatum which develops from the same mass as the basal nucleus including the surrounding tissue (*palaeostriatum augmentatum*).

The total palaeostriatum in man is represented by the globus pallidus and eventual vestiges of grey substance occurring in and mesially to the capsula interna (underneath the *fiss. neo-palaeostriatica*) including a vestige of grey substance which is a continuation of the latter and (as in birds) lies underneath the caudate nucleus: the *substantia palaeostriatica caudata*, which in some animals may be separated from the caudate nucleus by a continuation of the lamina medullaris externa and of the *fiss. neo-palaeostriatica*. The palaeostriatum arises entirely from the base of the forebrain near the anterior wall of the preoptic recess. It consequently is entirely telencephalic in character not of diencephalic origin as SPATZ¹⁾ supposes, though its cells in adult animals may be continuous with the ventral and peduncular cells of the thalamus and midbrain as I already pointed out in 1908²⁾.

The *neostriatum* (putamen and caudate nucleus) in mammals, arises as well from the base of the brain in front of the palaeostriatum (immediately behind the anterior olfactory ventricle) as from the adjacent pallium (ELL. SMITH). The partly pallial origin of the neostriatum (already supposed — but not proved — by WERNICKE(?), OBERSTEINER and KÖLLIKER) explains the fact that in many cortical affections of the brain frontal lobe chiefly) as general paralysis (MILLS, L. BOUMAN, FORSTER, GANS) also the neostriatum is often affected, more often than the palaeostriatum.

Whether the hyperstriatum superius of birds which arises *only* from the mantle is included in the neostriatum in mammals or not, cannot as yet be settled with certainty.

The possibility exists that it is represented in mammals and man by the *claustrum*, which also is a ventricular ingrowth of the pallium (a hypopallial product in the sense of ELL. SMITH, l.c.).

Difference must be made between a cortical ingrowth and a (hypo) pallial ingrowth. Both are formed in the mantle, but need not necessarily to be related, though a pallial ingrowth may be followed by a cortical ingrowth (as f.i. is seen with the amygdala).

1) Ueber Beziehungen zwischen der Substantia nigra des Mittelhirns und dem Globus pallidus des Linsenkerns. Erg. Heft zum Anat. Anzeiger Bnd. 55, 1922.

2) Weitere Mitteilungen ueber die Phylogenese des Corpus Striatum and des Thalamus, Anat. Anzeiger Bnd. XXX, 1908.

Geology. — “*On Tertiary Marine Deposits with fossil fishes from South Celebes*”. By Prof. H. A. BROUWER and Dr. L. F. DE BEAUFORT.

(Communicated at the meeting of January 27, 1923).

The Origin and the Age of the Deposits by Prof. H. A. BROUWER.

During the construction of a road near Patoenoeang Asoe E in South Celebes a fossil-fish, of which only the posterior part is preserved, was found at the surface of one of the detached blocks of limestone. Mr. A. HUISMAN, the engineer who supervised the road-construction, sent me this fragment some time ago, informing me that in spite of further examination of the locality, the anterior part of the fish had not been found. The block was found near Patoenoeang Asoe E, Section Maros, at the base of steep rocks, about 50 m. above sea-level.

The limestone splits easily along the plane of stratification and on further examination in my laboratory¹⁾, it was found to contain another fossil fish far more complete than the first. Both were studied by Dr. L. F. DE BEAUFORT. His results are given below.

The rock in which the *Clupea* and *Lutjanus* are embedded is a whitetolight brownish compact limestone, which resembles some types of lithographic limestone from the neighbourhood of Solnhofen and Eichstätt in Franken, which contain the numerous well-preserved upper-jurassic fossils, among which numerous fishes occur. Under the microscope compact limestone proves to be free from foraminifera or other organic remains.

As regards their conditions of origin the rocks of Celebes are also very much like the lithographic limestones of the Upper-Jura in Franken. The latter rocks are found to rest in shallow-basins in coarse, unstratified or rudely stratified limestones, which are reefs on a large scale; the interjacent depressions were filled up with stratified deposits,

Near Solnhofen etc. these lithographic beds contain various, beautifully preserved organic remains. Fresh- or brackish-water fossils are lacking, but remains of tracks of land-animals are of

¹⁾ By Dr. P. KRUIZINGA, conservator at Delft.

frequent occurrence in the formation, from which it may be inferred that the lagoons between the coral-islands and reefs temporarily emerged above the sea-level and were exposed to the air.

Similar relations prevailed in that part of South-Celebes, where the fossil-fishes have been found. From personal experience I know the limestones near the cascade of Bantimoeroeng not far from the locality Patoenoeang Asoe E, and the numerous authors, who have described other parts of South-West Celebes, all mention these limestones, which often rise abruptly with steep walls from the surrounding plains, presenting typical reefmasses of Tertiary age.

Numerous foraminifera are found in these reefrocks; but also corals and shells are found. Rocks of oolitic structure are also recorded. As well as the limestones, which contain the fossil fishes and perhaps represent a lagoon-deposit, these oolitic limestones show the characteristics of deposits in a sea with reefs and lagoons.

Regarding the precise age of these rocks data have been published, notably by BÜCKING¹⁾ and VERBEEK²⁾ and afterwards by 't HOEN³⁾. We now know that Eocene limestones with nummulites and discocyclines occur amongst these rocks, as well as Miocene limestones with lepidocyclines.

Up to now no account has been given of foraminiferal limestones from the immediate surroundings of the locality where the fossil fishes have been found, the nearest rocks that were examined are those near the cascade of Bantimoeroeng, east of Maros, which contain *Cycloclypeus* and *Alveolina*. The age of these rocks may be Upper-Eocene or Oligocene. We have stated already that with the Old-Tertiary rocks also Oligo-Miocene limestones occur at various places.

The age of the fossil-fishes cannot be established exactly, because they show a slight relation only to the fauna of other regions. The

Clupea (*Sardinella*) is undoubtedly closely related to recent species, the *Lutjanus* possibly so. This might induce us to decide on a recent age of the deposits. But among the many herrings, for instance, from the Tertiary of Europe and America, none are described as showing a closer relationship to *Sardinella*. Different climatic conditions may have been of influence on the distribution of the fishes in Tertiary

¹⁾ H. BÜCKING. Beiträge zur Geologie von Celebes. Samml. d. geol. Reichsmus. in Leiden. Ser. I. Bd. VII. I. 1902. p. 118.

²⁾ R. D. M. VERBEEK. Molukken Verslag. Jaarb. Mijnwezen 1908. Wetensch. Ged. p. 52.

³⁾ C. W. A. P. 't HOEN. Verslag over de resultaten van geol. mijnbouwk. verkenningen en opsporingen in Zuidwest-Celebes. Jaarb. Mijnwezen 1915. Verhandl. p. 244.

times. VERBEEK¹⁾ pointed to this in connection with the fact that Eocene fish-species of the Highlands of Padang differ from the Tertiary species of Europe, whereas they bear a close relationship to the species still living in the East-Indian Archipelago, so that they seem to be Miocene rather than Eocene. From this it seems to follow that the Tertiary fishes of the tropics are not suitable to determination of age, and the species here described could be of the same age as the Eocene or Oligo-Miocene reefs.

Finally we wish to point out that after this discovery of fossil fishes, about which only very little is known as yet in the East-Indian Archipelago, it may be expected that on closer inspection of the locality more fossils will be found. The lithographic limestone of Solnhofen is poor in fossils. That the remarkable fauna of this formation has gradually become known is due to the quarry-industry and to the special attention given to the occurrence of fossils.

Description of the fossil fishes by L. F. DE BEAUFORT.

Prof. H. A. BROUWER entrusted me with the study of two fish-fossils imbedded in tertiary limestone, which have been found during the construction of a road in South Celebes.

The smaller and more complete fossil can be recognized at once as a *Chupeid*. As the anterior part of the head as well as the pectorals, ventrals and anal are missing, a further determination would have been doubtful, if the scales had not been extraordinarily well preserved. The greater part of the scales show a number of small holes in their posterior part, whereas the anterior part possesses more or less distinct transversal grooves, which are interrupted in the middle. As far as I know such perforated scales have only been found in four closely related species of herrings, which inhabit the Indo-Australian Archipelago.

These species belong to the genus *Chupei* sensu latiore. Following BLEEKER, WEBER and I (Fishes of the Indo Australian Archipelago II, 1913, p. 68) have placed these species with a number of others, which however do not show the characteristics mentioned above, in the subgenus *Harengula*. TATE REGAN (Ann. Mag. Nat. Hist. (8) XIX, 1917, p. 377) has raised this subgenus to the rank of a genus and has separated from it as *Sardinella* those species, which differ from *Harengula*, besides in some other characteristics, also in the structure of the scales. In *Harengula* the transversal grooves of

¹⁾ R. D. M. VERBEEK. Topographische en Geologische Beschrijving van een gedeelte van Sumatra's Westkust. Batavia 1883, p. 355.

the scales are uninterrupted, in *Sardinella* they are interrupted in the middle. The recent species with perforated scales mentioned above, as well as my fossil, have also interrupted transversal grooves on the scales, and TATE REGAN ranges these recent species therefore under *Sardinella*. It is clear, that my fossil belongs to the same group. TATE REGAN does not mention the small perforation of the scales in his short description and, therefore, I do not know if such perforations occur in other species, which belong to *Sardinella* and which inhabit the Atlantic, Mediterranean, Black Sea, Indic and Pacific. From the foregoing however it will be clear, that the species with scales of this structure form a natural group, and that the fossil belongs to it, which I proceed to describe now as:

Clupea (Sardinella) brouweri n. sp.

The total length of the specimen cannot be ascertained, as the praeorbital part of the head is wanting. The vertebral column is also broken at different places and some of the vertebrae have been shifted over each other, or got loose from each other. I estimate the length to be 150 mm. It is also difficult to count the number of vertebrae. I think I can distinguish forty-two of them which is somewhat less than the numbers, given by TATE REGAN l.c. for *Sardinella*. DELSMAN (Bijdragen tot de Dierkunde, Afl. XXII, 1922, p. 29) records forty-five vertebrae in *Clupea jimbriata*, one of the species with perforated scales.

Of the head skeleton really only the opercles and a part of the orbitalia have been well preserved. The ventral part of the opercle shows delicate vertical stripes, caused by sensory-canals and which, although in a somewhat different form, are also present in *Clupea jimbriata* and *perforata*. The preoperculum also shows some sculpture. Under favourable light fine lines, radiating from one point, may be detected, which I do not find so well developed in recent species. The operculum is not quite twice as high as long.

The dorsal rays cannot be counted accurately as part of the scales of the back have shifted on that fin. I think I can distinguish fifteen of them. Neither is it possible to ascertain the exact position of the dorsal fin, as the vertebral column has been distorted, as mentioned above. The origin of the dorsal is situated about in the middle between snout and base of caudal and is placed above the twenty-seventh vertebra, counted from the caudal. The longest D. ray is about equal to the height of the operculum.

The whole of the ventral part of the fish is severely damaged.

Nothing can therefore be said about the ventrals and anal. Of the pectorals and the pectoral-girdle only some rudiments are present.

As said above, the scales are well preserved, although most of



a
Scales of *C. (S.) brouweri* $\times 2$.

b
Lutjanus spec. $\times 1\frac{1}{2}$.

them are dislocated and shifted in a dorsal direction. Some of them are even quite isolated, which facilitates however their examination.

In comparing the scales with those of related species, we discover the greatest likeness with the scales of *Chupea perforata* and *jim-*

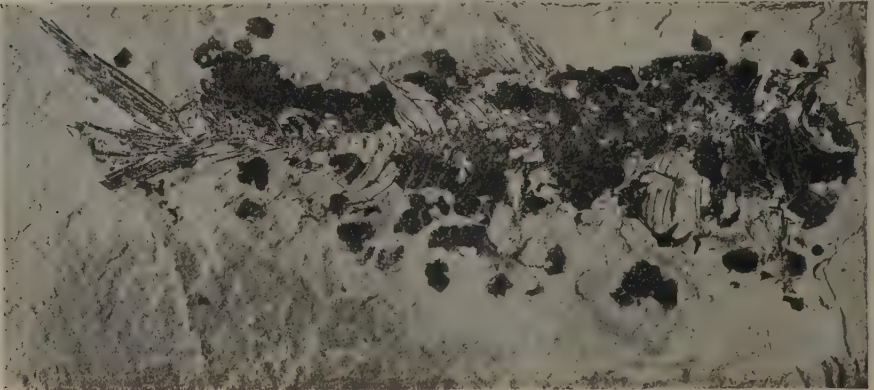
briata, which species, however, have holes of larger dimensions. Also the posterior border of the scales of these species is more ragged. The first transversal groove, which is not interrupted, is visible in all scales of my specimen, the following interrupted ones (about three in number) are specially developed on the caudal scales. For the rest the scales are practically smooth: only here and there some fine parallel stripes are visible.

As might be expected, keeled dorsal scutes are absent. The ventral ones are partly very well preserved, but all are dislocated and dispersed, so that their number cannot be made out. The dorsal prolongations of the ventral scutes or spines are beautifully conspicuous here and there. They seem to be shorter than in recent species. Possibly they are broken.

C. (S) brouweri shows the greatest resemblance to *C. jimbriata* and *perforata*, but differs in the sculpture of the opercles and in minor details of the scales.

The determination of the second fossil is less certain. It consists of the posterior part of a fish, which undoubtedly belongs to the *Perciformes*. The greater part of the caudal, all caudal vertebrae, some of the rib-bearing trunk-vertebrae, all pterygophores of the anal as well as those of the hinderpart of the dorsal are beautifully preserved in situ. The anal is broken.

The soft portion of the dorsal is intact, but only some spines of



Clupea (Sardinella) brouweri n. sp. $\times 1$.

the precedent part are preserved. The anterior part of the fish is wanting. The ventrals and part of the pelvic have been spared. Nothing else of the shoulder-girdle remains than the caudal part of the postcleithrum. The vertebrae of the trunk bear long parapo-

physes to which the ribs are attached. There are fourteen caudal vertebrae. The caudal has seventeen rays, two of which are probably simple. The dorsal has $x + 3$ spines and 17 soft rays, the anal has 11 and the ventrals have 6 rays.

Part of the scales are extremely well preserved. They are more or less rectangular, with a convex posterior border. From the centre about ten diverging grooves run to the anterior border. A great number of crowded parallel rows of extremely small flat spinelets run to the posterior border. For the rest, the surface of the scales is covered with delicate small lines, concentrically arranged round the middle of the scale and scalloped where they cross the grooves.

With these scanty particulars a further determination had to be tried. A first indication was the great difference between the number of soft dorsal and anal rays. In consequence a number of forms, in which the soft dorsal and the anal are of nearly equal length, could be excluded. Farther on the structure of the scales put me on the right track and brought me to the recent genus *Lutjanus*. It is true that most species of this genus have fewer D-rays than my specimen, but in some of them the number is about the same, f.i. in *Lutjanus sebae*, which species I, therefore, selected for closer comparison. A skeleton of lastnamed species shows so much likeness with my fossil, even in details, that I scarcely doubt that this too belongs to *Lutjanus*.

What can these two fossils now teach us about the age of the deposits, in which they were fossilized?

As far as I know, no other tertiary Teleosts are known from the Indo-Australian Archipelago, than those from a freshwater-deposit in the Padangsche Bovenlanden formerly described by GÜNTHER (Geol. Mag. (2) III, 1876). As far as I know, forms related to our fossils are lacking too in the tertiary fish-fauna of the neighbourhood. Neither amongst the tertiary fishes from Australia (CHAPMAN and PILCHARD, Pr. Roy. Soc. Victoria (2) XX, 1907) nor amongst those of Siam (ANDERSSON, Upsala Bull. Geol. Inst. XIII, 1916) a species of *Sardinella* or *Lutjanus* has been described. The Clupeid, recently described by JORDAN (Proc. Cal. Acad. of Sciences IX, 1919) from Japan, is not related to our specimen. It is even uncertain, if it is a Clupeid at all.

Among the many herrings, described from the tertiary of Europe and America, I do not know of any species, related to *Sardinella*. SMITH WOODWARD (Cat. Fossil Fish British Mus. IV, 1901, p. 152) gives the following description of the scales of *Clupea numidica*, from the Upper Miocene of Algeria: "Scales sometimes pitted in

their exposed portion." But "pitted" is different from "perforated". Besides *C. numidica* has 55 vertebrae.

No other fossil *Lutjanus* is known than a dubious species *Lutjanus hagari*, described by JORDAN and GILBERT (Stanford University Publications 1919) from the Miocene of California and which later on has been ranged by JORDAN in the related genus *Neomanis*.

Therefore, an opinion of the age of the two fish-fossils cannot be more than a guess. When we take in mind, that during the Miocene most of the recent genera were not yet in existence, as JORDAN has pointed out recently and when we remember, that the *Sardinella* is certainly related to recent species and the *Lutjanus* probably so, I feel on these grounds inclined to consider them not older than miocene.

Both fishes have been found in one stone, the dimensions of which are about $40 \times 20 \times 6$ cm. Moreover in the same stone some scales of other fish-species occur, which I do not venture to determine. This shows, that fish-rests are probably abundant in these layers. A further exploration would certainly be worth while, and could give us more solid information about the age and the character of these deposits.

Geology. — "*Fractures and Faults near the Surface of Moving Geanticlines. III. The Horizontal Movement of the Central-Atlantic Ridge*". By Prof. H. A. BROUWER.

(Communicated at the meeting of January 27, 1923).

Many explanations that have been given for tectonic structures are unsatisfactory on account of the geometrical treatment of the problems and a preference to vertical movements. The geometrical treatment draws attention to the change in position of parts of the earth's crust, while the velocity of the movement receives no further consideration. Because of the predilection for vertical movements we often explain the observed facts by vertical movements, until it is proved that faulting must have been effected in another direction.

In regions, which are not accessible to direct observation, i.e. the parts of the earth's crust covered by the sea, the existing morphology is explained by rising and by subsiding movements, while the factor time is neglected. Subsidence of continents and subsidence of "land-bridges" are common expressions in geological literature. Velocity and direction of the movement are hardly or not at all considered in these inadequate interpretations of dynamic phenomena. The reason is obvious, the forces causing the movement are unknown, and the velocity of the movement cannot be measured.

Another way of studying these problems is the comparative-tectonic method. Our object in this paper is to consider the results achieved by applying this method to the movement of a region, which is almost entirely covered by the sea, of which the morphology is known in broad outlines, and which is still moving, as we know from numerous earthquakes. It is the S-shaped ridge, of which the existence has been proved by numerous soundings and parts of which emerge from the sea, as e.g. the Azores and the islands of St. Paul and Tristan da Cunha. In previous papers¹⁾ we pointed to the significance of the bending-points of the horizontal projection of a geanticlinal axis for a judgment upon the horizontal movement of geanticlines. Transverse fractures, which may be more or less

¹⁾ These Proceedings XXIII, p. 570; XXV, p. 327.

H. A. BROUWER. The horizontal movement of geanticlines and the fractures near their surface. Journ. of Geology. 1921, XXIX, p. 560—577.

gaping are the surface expression of velocity-differences in a horizontal direction; horizontal transverse faults prevail at greater depth, while with increasing plasticity deformation takes place without fracture-movements. If these tectonic zones of different depths are all visible at the surface, they enable us to trace the movement for a considerable space of time, because then the different phases of the movement are observable. If the movements are still going on, the epicentra of earthquakes will be accumulated near the places with considerable velocity-differences and may be disposed along more or less transverse fractures. In this connection we point to the region in the neighbourhood of Sunda Strait between Java and Sumatra, to the earthquake lines near the bending-point between the Alps and the Carpathian mountains, to Cook-strait between the Northern and the Southern island of New Zealand and to many others.



- | | |
|-------------------------------|--------------------------------|
| Zuid Amerika = South America. | Romanche diep = Romanche Deep. |
| Afrika = Africa | Azoren = Azores |

Fig. 1. 2978 etc. Depths of the sea in meters on the Central-Atlantic Ridge.

If a submarine ridge has a bending-point, the strongly curved shape of the ridge may have been developed from an originally simpler form by velocity-differences in a horizontal direction. Where the velocity-differences are greatest, the epicentra of earthquakes will be numerous, and from an accumulation of epicentra near a

bending point it may be concluded that velocity differences in a horizontal direction are a characteristic of the present movement.

In the Central Atlantic Ridge there is a distinct bending-point between the island of St. Paul and the Romanche Deep, while quite close to it there is a zone of strong seismic activity. Further application of the comparative method would lead to the conclusion that the Central Atlantic Ridge is not only moving now, but has been moving for a long time, with velocity-differences in a horizontal direction. The tectonic structure of the ridge is not accessible to observation. However, there are indications that a further application of the comparative method is possible. The morphology is still little known, but the soundings have proved the existence of very great depths, viz. in the Romanche Deep, where a depth of 7370 m. has been sounded.

This depth has been considered as a remarkable phenomenon for the Atlantic Ocean. The situation *close to the bending-point* points to an origin such as already previously suggested by us with regard to abnormally deep straits near the bending-points of rows of islands. Just as is the case in Manipa Strait between Ceram and Boeroe. The Romanche Deep can be explained by difference in velocity of horizontal movements for neighbouring parts of the ridge along the axis.

We only find the results of the *differences* in velocity in a horizontal direction, the absolute horizontal movement cannot be inferred from the surface characters with the comparative method. We do not know whether the Central Atlantic Ridge originally had a more rectilinear form. Neither do we know whether the bending of the strong curve between the Azores and the island of St. Paul is still increasing, or whether the southern portion with Ascension and Tristan da Cunha is moving with less velocity than the northern in a western, or in an eastern direction, or whether it has become stationary now.

Many widely different views have been brought forward concerning the origin of the Central Atlantic Ridge. Some authors¹⁾ look upon it as a rising geanticline, as a mountain range in statu nascendi. Up to now these authors never considered the horizontal movements, which as evidenced before often are much more important than the vertical movements in rising geanticlines. Another explanation²⁾ has been afforded representing the ridge as the filling of an originally

¹⁾ E. HAUG, *Traité de Géologie* I, 1907, p. 164.

²⁾ A. WEGENER, *Die Entstehung der Kontinente und Ozeane*. 1922, p. 42.

narrow gaping fracture, which opened to the present Atlantic Ocean by horizontal movements of continental areas.

In either view regarding the origin of the ridge the movements can take place with velocity-differences in a horizontal direction. Other explanations, such as the ridge being of volcanic origin or the highest parts of a subsided continent (horst), do not consider horizontal movements. Vertical movements may occur and may have occurred in some places perhaps in an upward, in other places in a downward direction, and varying at different periods, because no movement of the earth's crust will have exactly a horizontal direction for a long time, just as it will never have exactly a vertical direction.

The comparative method does not enable us to trace out the movement of the Central Atlantic Ridge down from its earliest development. It proves, however, that the simple explanations by upward and downward vertical movements, which have been suggested, cannot be maintained.

Botany. — "*On stimulation in auxotonic movements*". By Prof.
J. M. JANSE. (Communicated by Prof. J. U. SCHOUTE).

(Communicated at the Meeting of January 27, 1923).

Many movements (curvatures) of very different plant-organs are caused by a change in the speed of growth on one side of the organ; collectively they are often called "auxotonic" movements. Various stimuli, among which those of gravitation and of light are by far the most important, may be the indirect cause of these movements; these stimuli are received locally and conducted to the growing zone in which the bending will afterwards take place.

The theory hitherto generally accepted was that the normal vertical longitudinal growth was a separate phenomenon, and that, for instance, the gravitation-stimulus appeared only after the plant-organ had been given a different position. In a recently published paper¹⁾, I expressed as my opinion that, on the contrary, the normal length-growth is also due to the gravitation-stimulus which by an increased growth of the cells equally on all sides would cause, for instance, the vertical growth of the main-axis and of the radicle. In this position there would even be the maximal stimulation corresponding to their maximal speed of growth in this position, which is experimentally demonstrated. The experiments carried out by WIESNER, MOLISCH²⁾ and CZAPEK³⁾ speak in favour of this theory; they showed that after the tip of the radicle had been cut off, the rate of growth diminished appreciably within the next 24 hours; this diminution would undoubtedly have been still more apparent if the observations had been recorded also during the ensuing days, because the growth during the first day must still have been influenced by the stimulus received before the amputation of the tip.

It is generally assumed that the stimulation by gravitation depends upon the pressure of the specifically heavier starch-grains (statoliths) upon the outer layer of the protoplast of certain cells (statocysts);

¹⁾ Reizwirkung bei Rektipetalität und bei senkrechtem Wachstum; *Jahrbücher für wissenschaftliche Botanik*, 1922, Bd. 61, p. 590.

²⁾ *Berichte d. d. bot. Gesellschaft*, 1883, Bd. 1, p. 362.

³⁾ *Jahrb. für wiss. Botan.*, 1895, Bd. 27, p. 246.

such a stimulus, however, as has already been demonstrated by NOLL (Heterogene Induction, 1892), can be the cause of a movement only if the sensitiveness of this outer layer is unequal at different parts. As the vertical position, in the said organs, was regarded as the one in which no stimulation took place, it was supposed that the part adjoining the lowest transverse wall was not sensitive. If, however, also the longitudinal growth be induced by the gravitation-stimulus, as we suppose here, that part would have to be on the contrary the most sensitive.

However this may be, it is sure that there must always exist a certain connection between the position of the place of the greatest (or least) sensitiveness in the statocyst and the direction of normal growth of each organ, so that this, for instance in the cells of the vertically-growing stem, must be found at a different place to those of a horizontally-growing rhizome, etc.

This ought to imply further that when an organ of itself changes its position, this should be preceded by a shifting of the outer layer of the protoplast inside the cell. The supposition of such a shifting of the outer layer would, however, be inconsistent with the general assumption that this layer is immovable, an hypothesis, it is true, but one which for other reasons, e. g. the existence of the plasmodesms, might be called probable. This inconsistency suggests the query as to whether it is not more probable to assume that the excitable portion of the statocyst forms a separate organ of the cell, which might then lie between the outer layer and the granular protoplasm, but quite independent from the former.

This protoplasmic part, which alone should be sensitive to the pressure of the starch-grains, might be termed the "*static apparatus*" and should be capable of shifting, consequent on some influence



The accompanying diagram represents, schematically, the supposed position of the "*static apparatus*" in the statocyst: Z = cell wall, H = outer layer, K = granular protoplasm, R = the static apparatus of which, M = the middle-field.

The unequal thickness of the static apparatus in the drawing serves *merely* to indicate the local difference in sensitiveness of this apparatus which should be greatest in the middle-field.

from inside or outside, without the outer layer of the protoplast being involved in this movement. Moreover this apparatus should have to be most sensitive in the middle-field, while this sensitiveness

should diminish towards the edges as represented in the accompanying sketch. The apparatus need not be present in cells which are insensible to stimulation.

The normal vertical position of the main axis and radicle would seem to imply that in these organs the middle-field lies against the basal transverse wall of the statocyst. But in a horizontally-growing rhizome, for instance, it ought to lie next to the lower longitudinal wall, for then only there would be maximal stimulus, accompanied by the maximum, equal all round, speed of growth, whereby the rhizome would keep its horizontal position.

If now a certain shifting of the static apparatus is required to produce a new position of equilibrium, then inversely we might deduce from the change in the position of equilibrium what shifting should have taken place in each separate case, but therefore it were necessary to know also in what part of the organ the static apparatus occurs. This shifting cannot be microscopically controlled, for the present at least, but if it should appear from the following lines that by assuming such a shifting we succeed in giving a simple explanation of widely different and often very complicated phenomena, this must favour our supposition of the presence of a *movable* excitable organ in the sensitive cell. It must be borne in mind, however, that it is therewith immaterial whether we think of an "static apparatus" as indicated above, or of the outer layer as a whole, provided this be but movable; in future we shall suppose the presence of a "static apparatus".

If it be possible by this means to explain why a plant-organ which has a certain position of equilibrium is able to keep this position during its growth, it does not, however explain the familiar phenomenon of an organ that is brought out of its equilibrium returning to this position, not only of its own accord, but also by shortest possible way; so a root, for instance, placed horizontally will curve downwards in a *vertical* plane until the tip points perpendicularly again. That this movement is of great advantage for the later development of the plant is of course no sufficient explanation of its cause, especially since the preparations for the movement are made long before the utility of the bending could be perceived by the plant. We should have to ask, therefore, why it is that a part of the plant makes a *useful* movement and how it comes that the new position is acquired by the shortest way.

This question which, as it seems to me, is proposed here for the first time so sharply, is connected so deeply with the more intimate life of the cell that it can not surprise that no entirely complete

answer to it can be given yet, but nevertheless we can endeavour to arrive a step nearer at its solution.

We shall confine ourselves now in the first place to the stimulus of gravitation.

We have thus supposed that the static apparatus of the statocyst lies in such a position that the middle-field, which forms its most sensitive part, adjoins the lower wall of the cell when the organ is in equilibrium, whatever this position may be. When this position be changed, if, for instance, a root be placed horizontally, the starch-grains which shift under the influence of gravitation, come into contact with the less sensitive border of the apparatus; if then, after some time, the tip bends downward, the starch-grains, again shifting, will gradually come into contact with the more and more sensitive parts of the apparatus till, when the tip stands vertical, they will have reached the most excitable place again; thus we see that the curving downward is accompanied by a continual increase of the stimulus and that the speed of this increase will be greatest when bending takes place in a vertical plane.

Could it be that this increase of the stimulus is the indirect cause of the bending and at the same time of the choice of the shortest way?

Of itself this "striving after the maximal stimulation", as we might term it, cannot be regarded in the plant as the direct cause of any movement, although it might later on be of aid in explaining it; nevertheless cases are known in which there exists a rather direct connection between this striving after an ever stronger stimulus and the movements.

So, for instance, in positive chemotaxis: if e.g. spermatozoides of ferns be placed in a weak solution of malic acid in which the concentration is unequal at different places, they will move towards the place of the strongest concentration, i. e. in the direction of the increasing concentration or stimulation.

It is known, regarding some of the senses of man and animals, such as the eye, the ear, and perhaps also of the static organ when the organism is at rest, that they adjust themselves automatically (reflectorily) to a stronger stimulus, i. e. that the same stimulus which causes the sense-perception also excites other nerves and through them certain muscles, which last thereby move the sense-organ in such a way that it receives then the strongest possible impression; thus here too we have the case of a movement with the aim of increasing the stimulation. If such a comparison with the plants were entirely justified, which could not be decided at present, we

might go further and state that, because the sense-perception is wanting in the plants, their bending might be compared with the purely reflectory movements of animals.

However, although it must be admitted that within the scope of physiology comparisons between plants and animals may be successfully drawn in many cases (as is probable especially with regard to stimulation, for the reason that in both groups of living organisms one and the same relation appears to exist between stimulus-intensity and stimulus-effect: the law of WEBER), this must be done always with the greatest caution. Bearing this in mind it nevertheless appears to me that the facts furnish us with sufficient reason to assume the striving on the part of the plant to receive the greatest possible stimulation by the quickest way as a supposition, just as we know this is the case with regard to positive chemotaxis. It must be left to later researches to reduce this striving after maximal stimulation to an actual cause of movement.

With the aid of a number of examples taken from the different groups of auxotonic movements, I now wish to demonstrate very shortly how simple the explanation of these phenomena becomes when we set forth from the assumptions mentioned above.

The different movements may be brought to certain groups according to the (supposed) position of the static apparatus and to the shifting which it should undergo.

A. STIMULATION BY GRAVITATION.

I. *Stationary position of the static apparatus.*

α. In the first place the static apparatus might lie against the lowest transverse wall of the statocyst. This should be so in the case of the vertically-growing main-root and main-axis, where the, maximal, stimulus should be the cause of the vertical growth of both by the equal lengthening of the cells allround.

If these same organs be placed in another position, e. g. horizontally, they will show positive (root) or negative (stem) geotropism. This we should now try to explain by the striving after a stronger stimulation. In the horizontal position the starch-grains press upon a part of the less sensitive border of the static apparatus; if they have to come into contact with the middle-field, the most sensitive part, the root will have to bend downwards, the stem, on the other hand, upwards. The explanation of these opposite movements requires therefore no new supposition; it follows from the circumstance that in the statocyst the middle-field in the case of the root lies against the transverse wall which is turned away from the growing-zone

(where the bending occurs) and in the case of the stem against the one that is turned towards it.

If in the centrifugal-experiment, the statolithes are moved outwards in the statocyst, then, for the same reason as given above, the stem must react by bending towards the centre, whereas the root will curve away from it.

b. Normal horizontally-growing plant-organs, such as rhizomes and some rootlets of epiphytes, can only maintain their position of equilibrium and continue growing in the same direction if the middle-field lies at the lowest longitudinal wall of the statocyst, for the same reason again that it is only in this position that the starch-grains will come in contact with this middle-field. Whether these organs also attain their quickest growth in this position has still to be investigated.

c. Besides lying against the transverse and longitudinal walls, the middle-field might also lie between the two, i. e. slanting; in such cases the organ should also exhibit a slanting position of equilibrium, the size of the angle it makes with the perpendicular depending upon the position of the apparatus with regard to the axis of the statocyst. This would explain the fixed position which the lateral branches and lateral roots of the first order always assume, and which is so different in different plants (cf. e.g. *Araucaria*, the common foliage trees, *Populus pyramidalis*).

II. *Variable position of the static apparatus.*

Various organs of plants undergo a change in equilibrium during their normal development which could be ascribed now to a shifting of the apparatus at a certain moment, that is to say, if it can be demonstrated that gravitation-stimulus or longitudinal growth plays a part in the phenomenon.

The shifting may take place either at a certain moment or be continuous; moreover it may occur autonomously or as a result of some outside cause. According to this we may distinguish the following cases:

a. The position of the apparatus changes, autonomously, at a certain moment.

During the germination of the seed of a twining plant the young stem is at first vertical, but very soon the summit assumes a more or less horizontal position and at the same time the twining commences. It is possible that this transition from negative to transversal geotropism were preceded by an autonomous displacement of the

static apparatus, whereby the middle-field is shifted from the lowest transverse wall to one of the longitudinal walls; by the bending of the stem this longitudinal wall would then become the lowest of the statocyst. If the apparatus in the cell shifts over 90° , the new position of the stem-tip will become exactly horizontal; if, on the contrary, it moves less, the stem-tip will, as is often the case, assume a corresponding upward slope.

Similar changes in position, as seen in many flowers before and after flowering, may be explained in an equally simple way. The flowers of *Narcissus*, for example, when in bud stand perfectly upright, but when about to open are practically horizontal, which again would point to a preceding shifting of the apparatus from the lower transverse wall to one of the longitudinal walls. In *Agapanthus* the same movement occurs, but goes farther on, because after fertilization the ovary bends still further downward; in this case a further shifting in the same direction should have taken place, by which ultimately the middle-field arrived at the apical transverse wall.

In all these movements the bending is accompanied by a distinct growth of flower- and of fruit-stalk. Amputation of the flower-bud will prevent these movements, for which reason it is assumed that the statocysts are situated in the ovary.

Other plants again exhibit the phenomenon that the peduncle which stands upright during bloom, after fertilization increases much in length and curves downward; this is most striking with those plants which bury their young fruit in the ground, e. g. *Trifolium subterraneum*, *Arachis*, etc.; here the shifting of the apparatus from the lowest transverse wall to the highest should take place in one phase.

In all these cases the change of position of the static apparatus is clearly a result of a separate new stimulation which is either the growth of the flower or the process of fertilization.

A shifting of the apparatus in a contrary sense should take place in those cases in which the tip of the sympodial rhizome bends vertically upwards for the purpose of producing leaves and flowers, because this upward curve would have to be preceded by a displacement from the lowest longitudinal wall to the basal transverse wall.

The best known instance of a particular curvation is that of the flowerstalk of *Papaver* (to which those of the peduncles of the inflorescences of *Tussilago Farfara* are closely connected), since there the movement has to take place before the flowering in one sense and after the fertilization in the opposite direction. VÖCHTING in 1882 succeeded in demonstrating that these movements are inti-

mately connected with the geotropic-stimulus both of the stalk and of the ovary, while the "rectipetality" should also play a part in it.

VÖCHTING gave the name of "rectipetality" to the phenomenon that a plant-organ, which has curved upon irritation, begins straightening itself out again as soon as the stimulation has ceased. This he regards as a separate quality of plant-organs since it further appeared that the straightening required no new stimulation. It seems to me, however, as I set forth also in my article quoted above, that rectipetality must be regarded rather as a consequence of the original stimulation which, being gradually conducted to the opposite side of the organ, causes a contrary curving.

In *Papaver* the young flowerbud stands upright on a short and vertical peduncle; soon, however, the rapidly growing stalk makes a curve of 180° , so that the bud now hangs inverted. In this position the peduncle continues to grow which takes place at the bend, without however the curve increasing, owing to the simultaneous tendency towards rectipetality, and so it seems as if the growth is limited entirely to the part below the bend. When the flower is fully formed, the bud rises again and this upright position is also retained by the fruit.

Amputation of the ovary only (inside the bud before it is full-grown) checks the growth of the stem, which then stretches straight out as much as possible; the cessation in the growth should be regarded as a result of the cessation of the gravitation-stimulus in the ovary, the straightening of the stalk as caused by the "rectipetality" which is then the result of the stimulus received before the ovary was cut off.

The peduncle as well as the bud is negatively geotropic; the static apparatus should thus again be supposed to lie against the basal transverse wall and this position should remain unchanged in the peduncle. The reason that the growing stalk bends over at an angle of 180° should be attributed to a shifting of the static apparatus in the statocysts of the ovary from the lowest transverse wall to the uppermost, while the erecting of the full-grown bud later on should be preceded by the opposite movement in the same cells.

This example shows well how simply these seemingly complicated movements can be explained upon our assumptions.

A last group will comprise the epinastic and hyponastic movements which are so common in plagiotropic organs.

These movements depend upon temporary inequalities in the speed of growth between the upper and lower surface of the organ (especially leaves), whereby the growth predominates now on one

side and then on the other. The reason of these changes in the speed of the growth is unknown, but, while a renewed research into these movements is highly desirable, it may be taken as fairly certain that, although all apparently similar, they are not so in reality, since they are evidently not all governed by the same stimuli. The influence of gravitation, for instance, can be demonstrated in many of them, so that for this reason and also because the movements depend entirely upon longitudinal growth, there is every reason to assume that statocysts are also present in these organs. With respect to the place where they occur in leaves in general, not much is known, and it would therefore be useless to make further premises regarding the shifting of the static apparatus before sufficient data on this point have been obtained.

Some movements, however, might already be explained in a similar way as above; so, for instance, the movements of the leaves in the unfolding buds of *Aesculus*; in the bud, and also as soon as it opens, the petiole and leaflets stand vertically upright, after which the leaflets make a downward bend of 180° at the joint (shifting of the apparatus from the basal to the apical transverse wall); finally the leaflets, as well as the petioles, take up an almost horizontal position (shifting of the apparatus in both to the undermost longitudinal wall).

In connection with the above I may refer to the very important though apparently almost totally forgotten observations of Hofmeister¹⁾, from which it would seem that the lateral growth of the leaves in the bud is frequently influenced by their vertical position so that the half of the leaf pointing upwards in the bud will grow faster than that pointing downwards. If these observations be correct they would form a further indication that statocysts are also present in the leaves and would thus be able to exercise an influence upon the growth of the cells. This would agree with my view, expressed above, namely that the static apparatus also governs the normal growth in length. We shall return later to the consideration of the influence of gravitation upon the normal position of the leaves, as this also should be connected with the influence of the static apparatus (page 184).

b. The position of the static apparatus is altered by external influences.

Sometimes an external influence leads to a change in the position of organs, as, for instance, amputation of the main-axis.

¹⁾ Allgemeine Morphologie der Gewächse. 1868, § 23.

If the terminal bud or a part of the main-axis be cut off, the lower lateral bud or lateral shoot will develop more strongly than it would otherwise have done, and will at the same time bend upwards until it assumes the position entirely, or almost, of the main-axis; amputation thus causes an accelerated growth as well as strong geotropic bending.

The absolute relation between the two, so striking here, is simply explained now by the circumstance that both are dependent upon stimulation of the static apparatus.

If, for instance, the almost horizontal lateral axis of *Araucaria*, after amputation of the terminal bud, gradually assumes a vertical position, this might have been preceded by shifting of the apparatus from the lowest longitudinal wall to the basal transverse wall, i. e. a shifting in the direction of the wound. TANGL.¹⁾ and NESTLER²⁾ now have demonstrated that the result of a wound is that in the neighbouring cells the protoplasm tends to accumulate in the direction of the wound; if it be that the static apparatus had a share in this shifting, this alone could be a reason for the appearance of the negative-geotropic movement.

It might be mentioned in this connection that, according to RICHTER³⁾, even a plant of so much more simple structure as *Chara*, shows the same phenomenon, namely, that after amputation of the terminal bud, the adjoining lateral branch grows out more quickly and bends sharply upwards.

Amputation of the radicle has not the same effect upon the side-roots of the first order; SACHS⁴⁾ has demonstrated that the lateral roots already present show no change in position, but that the after the amputation new formed lateral roots grow out in a more vertical direction, thus showing rather an influence upon the position of the apparatus in the newly formed cells instead of producing a shifting in those already present.

c. The static apparatus changes its position continually.

When the static apparatus is at rest in any organ, that organ assumes a certain position of equilibrium; in the case of a continual autonomic shifting, on the contrary, the organ will never arrive at a position of equilibrium and therefore never be at rest. Such ceaseless movements are known in the nutations and in the twining of plants.

¹⁾ Sitzungsber. der K. Akad. der Wissensch., Wien, 1 Abt., 1884. Bd. 90, p. 25.

²⁾ Ibidem, 1898, Bd. 107, p. 708.

³⁾ Flora, 1894, p. 416.

⁴⁾ Arbeiten des botan. Inst. zu Würzburg, Bd. 1, p. 622.

The nutations are now considered to be movements which take place without any stimulations, but their explanation is still wanting. As they depend, however, entirely upon longitudinal growth, they will be considered here to be induced by the stimulus of gravitation.

The least common case of nutation is seen in the peduncles of *Allium Porrum* which first hang over to one side, then straighten out and afterwards bend over to the other side, and so on. A slight displacement of the static apparatus might induce this movement; if, for instance, the middle-field lies against the basal transverse wall, the stalk, as we have seen, will assume a vertical position; should it then move slightly to one side, the stalk, in its effort to find the new equilibrium, would have to bend over to the same side; if the apparatus then moves back across the transverse wall and then shifts slightly to the opposite side the stalk would become straight and then also have to bend to that side, and so on. This autonomous shifting of the apparatus to and fro across the basal transverse wall would thus be sufficient to cause indirectly the "swinging nutation".

Much more frequent is the "rotating nutation", in which the tip of the stem moves as if over a conical surface; it may very well be imagined that this movement is brought about owing to the apparatus, as in the preceding instance, lying somewhat to the side of the transverse wall but is now pushed round in a circle, as it were, though in such a way that the middle-field remains always at the same distance from the centre of the transverse wall. The stem would then again have to follow the whole movement, always making the same angle with the perpendicular. The more the apparatus shifts, and keeps aside from the transverse wall during the nutation, the greater will be the angle at the apex of the cone described by the tip of the stem.

The twining movement was regarded by SACHS as being intimately connected with the rotating nutation, also because at that moment in both the influence of gravitation seemed to be excluded. Later on, however, it was demonstrated by NOEL that in the twining the effect of this stimulus showed itself as "lateral geotropism" whereby the gravitation stimulus brings forth the lateral movement of the apex by causing a difference in growth between the two opposite lateral sides of the stem.

This lateral geotropism thus causes the apex of the stem to swing round, with the tip in a more or less horizontal position, while at the same time the tip twists round its own axis in the opposite

direction. It appears to me that these movements may also be explained by an autonomous shifting of the static apparatus. We have seen (page 6) that the tip of the young stem which at first is vertical soon afterwards assumes an almost horizontal position, after which it begins to twine; this was then explained by a shifting of the apparatus from the basal transverse wall to one of the longitudinal walls which then by the bending became the lowest. And if this apparatus were displaced now again in the statocyst, so that it goes round the cell, but always keeping at one of the longitudinal walls, this would cause the tip twisting aback and at the same time its rotating in the horizontal plane, since this twisting could not take place without a simultaneous and equally rapid rotation (one turn for each circle described in the horizontal plane). This displacement should take place in the one direction in plants which twine to the left and in the opposite direction in plants which twine to the right. If the summit of the stem is not perfectly horizontal in rotating as often occurs, the apparatus should have to lie still at the longitudinal wall but somewhat shifted towards the basal transverse wall and should be carried round in this same position in the cell. .

It is worth noting in this connection that this displacement of the apparatus, and also the nature of the movement of the stem, agree largely with those described for the rotating nutation above-mentioned; for this reason, and because, in our opinion, *both* are to be regarded as dependent upon the gravitation-stimulus, the old supposition of SACHS is confirmed again, viz. that twining and rotating nutation are movements intimately connected with each other. The only difference would consist in the size of the apical angle of the cone described by the tip of the stem (which in twining plants may be as much as 180°) and thus, with regard to the static apparatus, in the distance, which exists continually during the shifting between the middle-field and the centre of the basal transverse wall.

This discussion, though necessarily too short, may however suffice to show that with the help of our theory it is possible to give even a simple explanation of the lateral geotropism.

A shifting of the apparatus back to the original position at the basal transverse wall would again lead to the negative geotropism which causes the stem to raise itself when the twining ceases and by which the convolutions are pressed against the support.

The twisting of the stem which can frequently be observed as an accompanying phenomenon and which probably also depends upon the gravitation, cannot be discussed here.

B. STIMULATION BY GRAVITATION AND LIGHT.

Many plant-organs curve under the influence of an unequal illumination, as this causes an inequality in the longitudinal growth at different sides of the organ (heliotropism). Since this depends thus entirely on increase in length, these movements must be regarded here as being brought about both by gravitation and by light.

It is known that light can cause certain movements of protoplasm: the swarm-spores move towards light (positive phototaxis), whereby, according to the experiments of ENGELMANN, it is the uncoloured portion of the swarm-spore which receives the stimulus; if green cells are exposed to the light after having been kept in the dark, the chlorophyl-grains undergo a definite change of position, but resume their original place when withdrawn from the light.

These reasons would already be sufficient to assume that the position of the static apparatus also can undergo the influence of light, but such an assumption will become still more probable when it can be shown by different examples that a similar shifting of the apparatus, i. e. *towards* the light, could furnish us with a rather simple explanation of very different familiar phenomena.

a. Positive and negative heliotropism.

In the vertical position of main-axis and radicle, as was said above, the middle-field of the static apparatus should lie against the basal transverse wall of the statocyst; if these organs receive light from the side, and the apparatus, as we have just supposed, moves towards the source of light, these organs can no longer be in rest, and they can find the new equilibrium, i. e. the starch-grains will come to rest again on the most sensitive middle-field of the apparatus, only if the stem moves towards the light, and the root on the contrary from the light; thus the familiar positive and negative heliotropic curvatures.

If the plant is replaced in the dark the organs return to their vertical position, from which we should have to infer that after cessation of the light-stimulus, the apparatus of itself returns to their former place at the basal transverse wall. Consequently this is the same thing observed with the chlorophyl-grains in the above-mentioned cases, namely, that they are brought out of their position of equilibrium by light and return to it when replaced in the dark. PFEFFER¹⁾ considers this a matter of course.

b. It is known that certain rhizomes react to light in such a

¹⁾ Pflanzenphysiologie, 1904, Vol. 2, p. 780.

manner that when their tip receives the light they acquire positive-geotropism and bend downward; when the tip pierces the ground again and is thus no longer illumined, the transverse-geotropism reappears

These movements too may be explained in the simplest way from our suppositions. In the normal rhizome, as we have seen (page 6), the static apparatus should lie against the lowest longitudinal wall; if, under the influence of light, the apparatus is displaced again towards the source of the light, i. e. in the direction of the apical transverse wall, the tip will have to bend downwards in order that the starch-grains may again reach the middle-field, and this is just the movement that we see the rhizome make. When again in the dark the apparatus, and therefore the rhizome too, will resume its former position, as in the preceding case.

c. The sleep-movements of leaves, as will be known, are influenced by light to such a degree that it was long believed that light alone was the cause of them. Later, however, exhaustive researches, in particular those of PFEFFER, showed that gravitation has also a share in them. This has been most clearly demonstrated for instance in the experiment with *Phaseolus*, in which the petioles of the two first leaves were secured during the day in their normal position, so that only the leaflets could make the sleep-movement. When the plant was then turned upside down, the nyctitropic movement took place at night, but showed exactly the reverse of what in the normal position occurred, i. e. in the light the leaves now stood vertically upright, whereas in the dark they were spread out horizontally. Thus, with respect to gravitation the leaves moved in the same direction as before, with regard to light however in a manner exactly contrary to the normal way, from which it is evident that it was the gravitation in the first place which governed the nyctitropic movement and determined the equilibrium of the leaf.

As practically nothing is known regarding the position of the statocysts in the leaves (see page 179), it is still difficult to express here any opinion with respect to the eventual shifting which the static apparatus might undergo here under the influence of light, the more so because there are so many varieties of nyctitropic movement. Important in this respect for an explanation in the sense as meant here, however, is the fact that it proved the presence of the principal auxiliary, namely the static apparatus itself, in leaves which show sleep-movement.

d. What has been written concerning the sleep-movements is really also applicable to the movements which cause the leaves to

assume their natural position; apparently they are influenced only by light, but here again the experiments of PFEFFER have shown that gravitation plays an important part; f. i. many leaves when brought away from their normal position can return to it in the dark, which evidently can be effected only through the medium of the gravitation-stimulus.

Here again the lack of data regarding the position of the statocysts in the leaves prevents us from prosecuting the research as to these movements in connection with our theory.

C. STIMULATION BY LIGHT ALONE.

Auxotonic curves are seldom caused by the light-stimulus alone; the instance of this most fully investigated is that of the "transversal heliotropism", whereby certain leaves place themselves perpendicular to the incident bundle of light. HABERLANDT¹⁾ endeavours to explain this movement by assuming that the middle-field of that portion of the outer layer that adjoins the lower wall of the sensitive epidermical cells is more sensitive to light than its surroundings. If now the leaf seeks to reach the desired position by the shortest way, this must be accompanied by the quickest increase in the intensity of the stimulus, exactly in the same way thus as was assumed above with respect to the stimulus of gravitation.

The stimulation of an sensitive organ causes everywhere a certain sensation or movement, whereby, however, the *nature* of the sensation or of the movement, is determined *solely* by the special properties of those parts of the organism which lie *outside* the perceiving sense-organ; consequently the nature of the stimulus can never exercise any influence whatever upon the effect that the organism shows.

If this conclusion should hold good for the plant too, as is very probable from the nature of the case, and if we also bear in mind that all auxotonic movements mentioned are executed in the same way, it would follow that it is sufficient for the plant to possess only one single sensitive organ for all these movements, induced by gravitation, by light or by both.

Therefore not even for transversal-heliotropism an exception should be made, for if we consider that a static apparatus without statoliths (starch-grains) could not be stimulated by gravitation but can nevertheless remain sensitive to light, it might very well be possible

¹⁾ Die Lichtsinnesorgane der Laubblätter, 1905, p. 127.

that the mentioned apparatus of HABERLANDT, sensitive to light and in which the starch-grains are always lacking, might be identical with our static apparatus (provided that in this case it should be unmovable), for both exercise exactly the same influence upon the growth-phenomena in the joints, etc.

These expositions might serve to show that the hypothesis of the presence of a movable "static apparatus" in the statocyst affords such a great advantage in the consideration and the grouping of the mentioned auxotonic movements, that it is entitled to be duly regarded as a working-hypothesis of sufficient foundation and further that there is probably in plants (and in animals?) a general striving towards an increase of stimulation which might later serve to find a further explanation of how these appropriate movements be brought about.

Leyden, January 23, 1923.

Mathematics. — “*On the Points of Continuity of Functions*”. By
 Prof. J. WOLFF. (Communicated by Prof. HENDRIK DE VRIES).

(Communicated at the meeting of February 24, 1923).

Let $f(P)$ be a function of the coordinates of a point P in a space with an arbitrary number of dimensions. The points where f is continuous, form an *inner limiting set*, i.e. the intersection of an enumerable set of open sets of points Ω_n , where we may assume that Ω_{n+1} is a part of Ω_n for any n . For the points, where the function oscillates less than $\frac{1}{n}$, form an open set Ω_n because the oscillation is an upper semi-continuous function. The set of the points of continuity is the intersection of all Ω_n , $n = 1, 2, 3, \dots$. YOUNG ¹⁾ has shown that to any inner limiting set E given in a linear interval, there belongs a function in that interval which is continuous in the points of E and discontinuous in any other point. We shall give here a simple proof, which is directly valid for spaces of any number of dimensions.

1. Let a set of points E be given as the intersection of an enumerable set of open sets Ω_n , where Ω_{n+1} is a part of (or coincides with) Ω_n .

We define $f(P)$ for any point of space in the following way: in the first place $f(P) = 0$ if P lies in E . Now let P be a point not lying in E , n_P the least value of n for which Ω_n does not contain the point P .

We put

$$f(P) = \frac{\psi(P)}{n_P} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\psi(P)$ is the function which in the points of space of which all the coordinates are rational, is equal to 1, in any other point of space equal to -1 .

We may say that (1) holds also good for the points of E , if there we assume $n_P = \infty$.

2. Now we shall show, that $f(P)$ is continuous in the points of E and discontinuous outside them.

¹⁾ W. H. YOUNG. Wiener Sitzungsber., vol 112, Abt. II^a, p. 1307.

Let us first assume that P belongs to E . In this case $f(P) = 0$. If ε be an arbitrary positive number, we may choose the natural number v in such a way that

$$\frac{1}{v} < \varepsilon \quad . \quad . \quad . \quad . \quad . \quad (2).$$

As P lies in Ω_v and Ω_v is open, there exists a region U round P which lies also in Ω_v . For any point Q of U we have therefore $n_Q > v$, so that according to (1) and (2)

$$|f(Q)| < \varepsilon,$$

Hence f is continuous in any point of E .

Let us now assume P to lie in the complement of E . If P is not an limiting-point of Ω_{nP} , it has a neighbourhood U which has no point in common with Ω_{nP} and which lies in Ω_{nP-1} . For any point Q of U we have in this case $n_Q = n_P$. Hence

$$|f(Q)| = |f(P)|$$

As the points where f is positive as well as the points where f is negative, lie everywhere dense on U , the oscillation of f in P is equal to $2|f(P)|$.

If however P is an limiting-point of Ω_{nP} , every neighbourhood U of P contains a part of Ω_{nP} . For any point of that part $n_P > n_Q$, hence

$$\left| f(Q) - f(P) \right| \geq \frac{1}{n_P} - \frac{1}{n_P + 1} \quad . \quad . \quad . \quad (3)$$

As the points Q for which the inequality (3) holds good, have P for a limiting-point, P is a point of discontinuity of f . Herewith the theorem has been entirely proved.

Mathematics. — “*Inner Limiting Sets*”. By Prof. J. WOLFF.
(Communicated by Prof. HENDRIK DE VRIES).

(Communicated at the meeting of February 24, 1923).

HOBSON has been the first to prove the following theorem:¹⁾

An enumerable set of points which has no part that is dense in itself, is an inner limiting set, i.e. the common part of an enumerable set of open sets each of which we may assume to contain the following one.

BROUWER has given an extremely short proof, but just as HOBSON he makes use of the transfinite ordinal numbers²⁾.

In the proof which follows here, no use is made of these numbers.

1. If E_1, E_2, \dots are inner limiting sets, if further each E_k is a part of an open set Ω_k , while no two Ω_k have any points in common, also the sum $E_1 + E_2 + \dots$ is an inner limiting set.

For we may write:

$$E_k = \Omega_{k1} \Omega_{k2} \dots, \quad k = 1, 2, \dots$$

which means that E_k is the set of points lying in Ω_{ki} for every i . The Ω_{ki} are open sets of which we may assume that they all lie in Ω_k . The set

$$(\Omega_{11} + \Omega_{21} + \dots)(\Omega_{12} + \Omega_{22} + \dots) \dots$$

contains $E_1 + E_2 + \dots$, but no point outside them, as $\Omega_{ki} \Omega_{lj} = 0$ for $k \neq l$. Now the auxiliary theorem has been proved.

2. We call a set of points E an inner limiting set in a point P if there exists an open set containing this point, so that the part of E lying in this set is an inner limiting set. This holds also good for the part of E lying in an arbitrary open set which is a part of the above mentioned one.

3. If an enumerable set E is an inner limiting set in each of its points, E is an inner limiting set.

We call the points of E : P_1, P_2, \dots

¹⁾ Proc. London M.S. (2) 2, p. 316—323.

²⁾ These Proceedings, Vol. XVIII p. 48 (1915).

Round P_k as centre we take an interval I_k (a quadrangle, a cube, etc. according to the number of dimensions of the space in which E is given), so that $E I_k$ is an inner limiting set, taking care that the boundary of I_k contains no point of E , which is possible on account of E being enumerable.

By I_k we understand the open interval, by \bar{I}_k we shall indicate the closed one, by an accent, the complement of a set. Now

$$E = E I_1 + E I_2 (\bar{I}_1)' + E I_3 (\bar{I}_1)' (\bar{I}_2)' + \dots$$

From N°. 1 there follows now immediately that E is an inner limiting set.

4. Let E be enumerable and not an inner limiting set. In this case according to N°. 3 the set D of the points E in which E is not an inner limiting set, is not empty. Let P be a point of D and I an interval with P as centre. $E I$ is according to N°. 2 not an inner limiting set, hence neither is $E I - P$; according to N°. 3, $E I - P$ contains a point Q in which $E I - P$ is not an inner limiting set, hence E is not an inner limiting set in Q , so that Q lies in D . From this there follows that D is dense in itself and from that the theorem which was to be proved.

Petrography. — "*On the Rocks of Doormantop in Central New Guinea*". By W. F. GISOLF. (Communicated by Prof. G. A. F. MOLENGRAAFF).

(Communicated at the meeting of February 24, 1923).

During a causerie about New Guinea, delivered at Batavia, Dr. H. J. LAM of Buitenzorg, at a meeting of the "Koninklijke Natuurkundige Vereeniging", showed a sample of a rock from Doormantop, which directly engrossed my attention to such an extent that I asked him to leave it to me for examination. He readily did so. Afterwards he furnished me with more samples of the same material, for which kindness I hereby tender him my best thanks. The geologist of the Mamberamo-expedition Dr. P. F. HUBRECHT, was staying in East-Java at that time, and was not in a position, within the first ten months, to send me any material. However, when asked, he did not object to an examination of the samples nor to publication of the results. I have much pleasure in thanking him also for his kindness.

The first samples that came to hand, present a schistose structure, chiefly due to parallel bands of magnetite; they are of a dark green colour, with a thin light-brown non-detached weathered crust of a cavernous appearance, on either side a relatively considerable quantity of magnetite reveals itself in non-crystallized masses; the erosion has spared the magnetite, so that it projects $\frac{1}{2}$, — 1 centim from the rock. A blow of a hammer made the rock split along the magnetite, thus effecting the first separation between the rock and the ore.

Some slides were made of the part from which the magnetite had been removed as much as possible. Under the microscope the rock proved to consist of magnetite with fresh olivine and a colourless, lath-shaped mineral of moderate refringence and very weak birefringence. The structure is slightly varying, the olivine now encloses the colourless mineral, now it mingles with it as if they were crystallized out simultaneously; the magnetite encloses the colourless mineral and occurs xenomorphic in the aggregate olivine-unknown material. The magnetite is polarimagnetic. A little apatite presumably occurs.

The olivine, which extinguishes undulatorily, but not to such a degree as is the case in most peridotites, looks very fresh and is absolutely free from weathering. The apparent weathering in the crust appears to be merely a brown colouring; serpentinization as an effect of atmospheric influence is absent. In another slide the refractive indices were, by the immersion-method, fixed at 1.66 and 1.70, after the Canada balsam had been carefully extracted by the use of xylol. The thickness of the slide was $\frac{1}{8}$ mm. (measured by detaching the slide and fixing it with tallow vertically on the object glass); the highest interference-colour observed was green 3rd order, making $\gamma - \alpha$ about 0.04, which agrees with the determination of the indices. The observation perpendicular to an optical axis in convergent light revealed on rotation of the table a slightly curving beam, at which the optically positive sign and a large axial angle could be established. Presumably one has to do here with a ferromagnesium olivine with about 10 % to 12½ % iron-olivine and 90 % to 87½ % magnesium-olivine (See DOELTER Handbuch der Mineralchemie II, I p. 16).

The colourless mineral, however, caused most trouble in its determination. Long as well as short laths occur; quadratic sections are lacking; the birefringence is low, sometimes next to zero; in one and the same lath the double refraction is not always the same, but varies, without attaining however, the so called „Pflock-structure”. All the laths show straight extinction; the elongation is invariably positive; cleavage lines run lengthwise through the crystal, especially in the middle and parallel to the outline. It was very difficult to obtain an interference figure. Therefore it was surmised that the mineral might belong to the melilite group, but this surmise proved to be untenable, as it was in no way supported by further microchemical and optical testing.

For this reason I applied to Dr. LAM for more material. This additional supply enabled me to identify the mineral. The coverglass was taken from all the slides, which were rinsed repeatedly with xylol, in order to remove any trace of Canada-balsam from the margins of the slide before being examined by the immersion-method. The refractive index appeared to be 1.58.

Being treated with hydrochloric acid and washed cautiously, gelatination ensued; when moistened with fuchsin and again washed carefully, the olivine as well as the unknown mineral under consideration appeared to be gelatinized. To make sure that the silicic acid gel of the olivine had not spread over the unknown mineral as well and might thereby be misleading, the whole procedure was repeated

and brought to light that the mineral under consideration gelatinized sooner than the olivine. In the liquid that had been collected microchemically the presence of calcium could not be detected.

Finally each individual lath was examined conoscopically; thus I succeeded in establishing in several of them that the mineral is biaxial, and that the axial plane is always perpendicular to the longer axis, the elongation being always positive. This is possible only if the mineral is developed into flakes perpendicular to the optical A-axis; it thus became more and more probable that the mineral could be rhombic. If so we must have to do with antigorite in its primary form.

The idea of a secondary postmagmatic genesis should be dropped altogether, the antigorite laths traverse freely from one olivine-crystal into another; subsequently they form with them as it were a eutectic crystallization and ultimately become the predominant mineral (See the microphoto fig. 1 and 2). All this applies to the material rich in large magnetite masses. Other material, finer grained

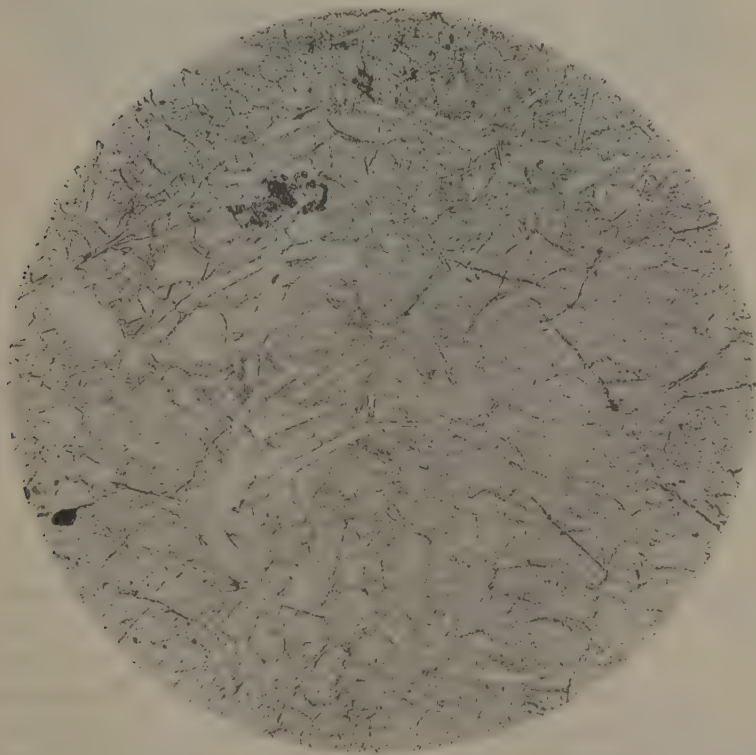


Fig. 1. Primary antigorite in olivine.

and poorer in large magnetite masses appeared to my surprise to be made up of the ordinary antigorite-serpentine with so-called olivine-

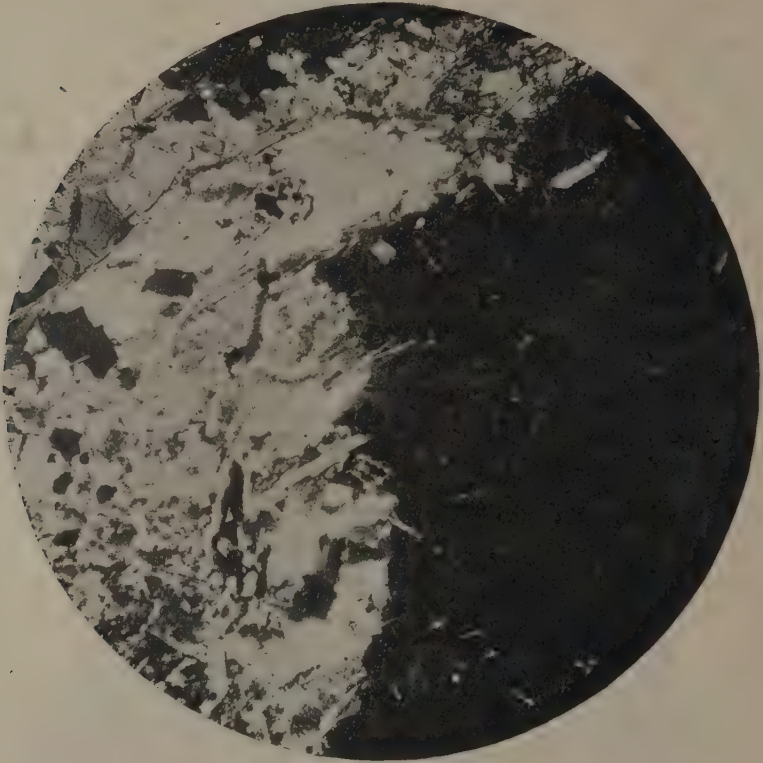


Fig. 2. Same as fig. 1, in polarized light with crossed Nicols.

rests. It appears to me that the cross-grained structure in those serpentines and the fine grain are caused by rapid crystallization. Also in these specimens the olivine is quite fresh.

The sequence of crystallization as manifested in the slides is the following:

Antigorite; antigorite-magnetite; antigorite-olivine-magnetite.

To all appearance the latter combination is a eutecticum, although it is not impossible that the magnetite is resorbed later.

Since the rock lies near the surface, the conditions for serpentinization by meteoric agencies have been favourable. However, of this the rock does not present any recognizable trace. The question, therefore, urges itself upon us whether the serpentine might perhaps be always of a magmatic origin, at all events not a product of weathering.

Now, as to the genesis of this rock we may broach the supposi-

tion (until the system $\text{H}_2\text{O}-\text{MgO}-\text{FeO}-\text{Fe}_3\text{O}_4-\text{SiO}_2$ shall be investigated) that the magma, from which this rock originated, crystallized under such a pressure that the gaseous components (notably watervapour) could not escape and consequently were taken up into the rock substance from the very beginning of the crystallization, thus occasioning a primary origin of serpentine. Putting it chemically ¹⁾: the crystallization begins in the serpentine-field and terminates in a point serpentine-olivine (magnetite?), which is perhaps located close to the connecting line olivine-serpentine (because Fe_3O_4 takes up only little space in an olivine-serpentine structure). In the case of eutecticum this point will be found on the same side as Fe_3O_4 , and in that of resorption in the common field of serpentine-olivine.

It may be suspected that in other peridotites, in which olivine crystallized first, the said pressure was less, so that, indeed, the gases could escape at the beginning of the crystallization, but were taken up again afterwards at the final crystallization, so that in similar cases serpentization of olivine might be considered as an apomagmatic (hydrothermal) process. Expressing it chemically: the crystallization then begins in the olivine field; on increasing pressure the stability field of the olivine is subsequently left for that of the serpentine. The consequent segregation of magnetite is self-evident after what has been said before. Magnesium is also set free for the forming of periclase or picotite or magnesite. As the gases move upwards it is obvious that serpentization will occur chiefly in the upper zones of peridotite-masses and on rents in the solidifying and consequently shrinking peridotite-masses.

Erosion being a downward process, first the marginal portions are laid bare, so that in the field the serpentine will in many cases be found prior to the olivine, which fact, I think has lent support to the erroneous but current view that serpentine is a weathering product.

After the foregoing had been written (August 1922), the chemical analysis came to hand (Dec. 29).

Of a sample freed as much as possible from magnetite an analysis was made at the Head Office of the Mining Department by Mr. A. TER BRAAKE and Mr. G. J. WALLY. The loss of water has been

¹⁾ To simplify matters it has been assumed that the serpentine and olivine are very definite compounds, which is not the case, of course. For the thermal-pressure-diagram of the five-substance-system a six or seven-dimensional space would have to be used, which would not facilitate the conception.

determined at 100° and at 200°, the latter temperature was maintained for three days, viz. until the weight remained constant.

The result of the analysis is:

SiO ₂	40.46
MgO	40.20
FeO	7.69 (determined as Fe ₂ O ₃)
Al ₂ O ₃	4.12
H ₂ O (100°)	6.14
H ₂ O (200°)	1.60
	<hr/> 100.21

CaO, MnO, Cr₂O₃, NiO are absent, as well as P₂O₅; no estimation could be made of K₂O and Na₂O, because in Java platinum chlorid at that time could not be obtained. In the determination of the iron-amount FeO and Fe₂O₃ were not estimated separately; it is likely, however, that they are both present.

It is evident that the chemical analysis fully confirms the microscopical examination.

Judging from the analysis also pyroxene is probably present, either separately as in so many peridotites, or in solid solution with, or as a component of the serpentine.

Presumably the latter is the case, since pyroxene has not been found in any of the slides.

It must be remembered that DAUBRÉE¹⁾ already succeeded in demonstrating that at a high temperature serpentine passes after melting into olivine + enstatite, while water escapes:



When leaving aside the watervapour, this case is merely a subdivision of the system MgO—SiO₂, which has been examined by ANDERSEN and BOWEN. DAUBRÉE's experience²⁾ is in complete harmony with their results; so for instance from a mixture of the system Mg₂SiO₄—MgSiO₃, on cooling first Mg₂SiO₄ crystallizes, which at 1557° begins to react with the solution, in consequence of which MgSiO₃ is formed which is precipitated on the surface of the olivine; at the same time the solution becomes richer in silica, so that ultimately SiO₂ can be set free; as DAUBRÉE added magnesia he did not obtain cristobalite. In the light of later experiments his

¹⁾ DOELTER failed in this experiment. Still, it is worth while to peruse DAUBRÉE's carefully described experiments.

²⁾ Comptes Rendus 1866, I, p. 660.

observations are correct; e. g. „Des aiguilles d'enstatite y sont fréquemment disséminées ou en recouvrent la surface” (i. e. of the olivine, obtained through smelting of serpentine with the addition of magnesia) and again in case he did not add magnesia: „le (péridot) se montre en moindre proportion que dans les fusions faites en présence de la magnésie.

Now since most peridotites (with the exception of dunites) consist of olivine and pyroxene, consequently of orthosilicate and metasilicate, we may venture to bring the primary and the secondary serpentinization into one focus. For a general theoretical treatment of the case the knowledge of the thermal pressure-diagram of the system $H_2O-MgO-SiO_2$ would be a first step.¹⁾ Needless to say, that this diagram will become very complicate owing to the great difference in volatility of the components.

From the foregoing it is evident, however, that under a pressure of one atmosphere serpentine is unstable; it would be worth while to repeat the experiment of DAUBRÉE in watervapours of various tensions in order to establish the limit of stability of serpentine. Now if we are right in supposing that olivine and pyroxene are not stable at high pressure and in the presence of watervapour, but that they are transformed into serpentine, the former with liberation of MgO ;²⁾ the latter with precipitation of silica, serpentinization may be accounted for as follows:

1. If the pressure is high enough serpentine crystallizes first from a magma, which is composed of $x Mg_2SiO_4$, $y MgSiO_3$, $z H_2O$; at a lower pressure the crystallization begins with olivine.

2. When olivine and (or) pyroxene are segregated, the volatile components congregate in the upper zone of the batholite, which may give rise to a high tension, in case they have no opportunity to escape; thus the field of stability of the olivine and (or) the pyroxene is abandoned, and that of serpentine is attained, after which serpentinization of olivine and pyroxene commences, occasionally with a residue of MgO (Fe_2O_3) or (and) silica; while in most cases MgO is present as magnesite.

Already DAUBRÉE acknowledged: „Rien ne prouve d'ailleurs que l'hydratation qui s'est produite dans la transformation des roches de péridot en serpentine ait été opérée par les agents de la surface du globe”.

¹⁾ See e. g. H. E. BOEKE, Grundlagen d. phys. chem. Petrographie, p. 179.

²⁾ E. g. as magnesite, because the component carbondioxyde is always present. Many serpentine deposits in fact contain magnesite and quartz.

It scarcely needs to be pointed out that, under the influence of the volatile components of a later intruded igneous rock, a peridotite mass may also be altered into serpentine.

Let it be recalled here that a résumé of the olivine-serpentine problem has been brought forward by W. N. BENSON (Origin of Serpentine, American Journal of Science 46 p. 693, 1918). It is to be regretted, however, that the problem has not been dealt with from a physico-chemical point of view.

Finally I beg to use this opportunity to thank Mr. A. C. DE JONGH, Director of the Research Committee of the Mining Department, for his willingness to have the analysis and the slides made in his laboratory.

From the above it may be inferred that many difficulties have stood in my way by the insufficiency of my laboratory-equipment. It is to be hoped that the Government of the Netherlands East Indies, which are so extremely rich in occurrences of beautiful rocks, may, at no distant date, take measures for the building of a well-equipped petrographic laboratory.

Wolvevreden, Aug./Dec. 1922.

Palaeontology. — “*New Findings of Pliocene and Pleistocene Mammals in Noord Brabant, and their Geological Significance*”.

By I. SWEMLE and Prof. L. RUTTEN. (Communicated by Prof. G. A. F. MOLENGRAAFF).

(Communicated at the meeting of February 24, 1923).

In the past year the Geological Institute of Utrecht obtained, partly through mediation of the Geological Survey, partly from the Government Bureau for Watersupply, some remains of fossil mammals originating from the southern and the western part of Noord-Brabant, one of the southern provinces of Holland, a district which up to the present has yielded very little in this respect. As we know, representatives of the young-diluvial fauna have been found in some localities of Noord Brabant, e.g. *Bos primigenius* Boj. near Den Bosch, *Elephas primigenius* Blumenb. near Acht, *Rhinoceros antiquitatis* Blum., in Hollandsch Diep. It is noteworthy, however, that in two places, near Oosterhout in the northwest, and near Westerhoven in the south of the province, remains have been recognized of a pliocene fauna, viz. *Elephas meridionalis* Nesti, and *Rhinoceros etruscus* Falc.¹⁾

Now, part of the remains, detected last year, have been derived from the zone between Oosterhout and Westerhoven. Three findings of mammals, belonging to the young diluvial fauna occurred in the vicinity of Esbeek S. S. E. of Tilburg, viz. a molar from *Elephas primigenius* Blum., found by Mr. SISSINGH on the premises of the clay-pit to the north of Esbeek, under a deposit of loam at a depth of three meters; three molars from *Rhinoceros antiquitatis* Blum., unearthed from a depth of 2½ m. in peat-bearing layers of clay, during the construction of the lock in the Wilhelmina Canal near Diessen, when the canal was being dug, and a molar from *Equus Caballus* L., found during the construction of the same canal to the east of the Diessen-lock at a depth of 3½ M.²⁾

The above fossils are not highly remarkable in themselves. The Molar from *Elephas primigenius* is a M III, sup. sin., on which

¹⁾ L. RUTTEN. Die Diluvialen Säugetiere der Niederlande, 1909.

²⁾ Far more eastward, viz. near Breugel on the Dommel, a fragment of a horn of *Bos Primigenius* Boj. was found, with which the Utrecht Geol. Inst. was presented last year.

— $19\frac{1}{2} \times$ are still visible on $215 \times 100 \times 160$ mM.¹⁾ The extremely thin lamellae and the slight thickness of the enamel prove conclusively that the tooth is to be referred to *El. primigenius*; it is remarkable however, that the enamel bands are finely folded which occurs only rarely in *El. primigenius*. The remains of *Rhinoceros antiquitatis* are three successive teeth, of one set of the right lower-jaw, viz. P 3, M 1, and M 2. They are but little worn down and have therefore belonged to a young animal; they must undoubtedly be referred to *Rh. antiquitatis*; the very thick enamel, the distinct striae of the enamel bands, the deep depressions and the trifling convexity of the teeth, all point in the same direction, while for the rest the teeth are almost quite similar to a set pictured by J. BRANDT²⁾. The tooth from *Equis caballus* is also a M of the lower-jaw.

From Oosterhout, however, where already previously teeth and bones from *Elephas meridionalis* Nesti³⁾ had been found in a superficial layer of loam, in a locality not precisely indicated, remains of bones and fragments of teeth were also sent to us, that belonged to this species. They were met with at a depth of 34.75 M. below Amsterdam-level in the first of five borings executed for the Water-company of Western Noord-Brabant. The wells are situated to the left of the road from Breda to Oosterhout on the Vraggel moor.

The bones from the well cannot be further determined, but a fragment of a tooth, most likely the posterior part of a M. 1 sup. sin. is distinctly indicative of *Elephas meridionalis*. It presents $3 \times$ with a length of $7\frac{1}{2}$ and a breadth of 8-9 centims, while the height minus the root is about 8 cM. The fragment was not chewed down, but was sawn, in order to get an opportunity of studying its structure.

Indicative of *El. meridionalis* are: 1° the extraordinary thickness of the lamellae, which appears already from the lamellae-formula; 2° the extreme thickness of the enamel (up to 4 mM); 3° the large breadth and the small height of the tooth; 4° the way in which the chewing-figures originate, namely through fusion of the four annuli (see figure).

Not only do we recognize in this fragment all the characteristics of *El. meridionalis*, but those characteristics even become prominent in the extraordinary thickness of the lamellae and the enamel.

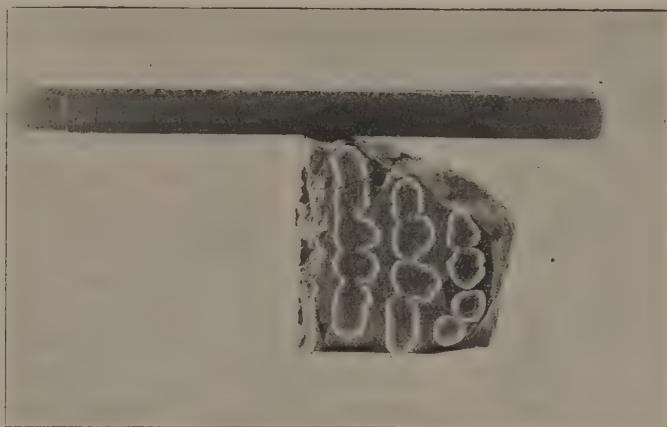
Dr. J. STEENHUIS kindly wrote us that the Geological Survey

¹⁾ H. POHLIG. Nova Acta Acad. Car. Leop. 53, p. 251.

²⁾ J. BRANDT. Mém. Acad. St. Pétersbourg. 1849. T. XI.

³⁾ L. RUTTEN. Die Diluvialen Säugetiere der Niederlande. 1909.

parallelized the part of the bore in which the tooth-fragment had been found, with the clay of Tegelen, which may be referred to the youngest pliocene or the oldest pleistocene. The tooth-fragment



corroborates this parallelism, for of late *Elephas meridionalis* has also been found near Tegelen ¹⁾.

From the fact that the previous discoveries of *El. meridionalis* were made near Oosterhout in a loam-quarry, near the surface, it may be concluded that in this part of Noord Brabant the pliocene rises locally to the surface. In the Annexes 11 and 13 of the "Final Report of the Government Exploration of Minerals", a fault running N. 40° W is marked West of Tilburg, which, however, in Annex 11 is drawn 2 KM. farther to the east than in Annex 13. To the north-east of this fault the soil has considerably subsided, as indicated on the sketch map; to the south-east the subsidence is less marked. When mapping the finding-places of the pliocene or the old pleistocene fauna (Westerhoven and Oosterhout), it will be noted that they fall to the east of the fault, as indicated in Annex 13, while Westerhoven would also lie within the trough, when assuming the course of the fault as marked in Annex 11. It is clear, however, that the pliocene can be expected near the surface only in the least subsided region, so that it is certain that the above-named fault — marked on the map only as a "suspected" fault — must be shifted more eastward. In that case, however, the locality of *El. primigenius* near Esbeek falls certainly to the west

¹⁾ S. RICHARZ. Centralbl. f. Miner. Geol. u. Pal. 1921 p. 664—669; id. Stadt Gottes 1921/22. Heft III.

of the fault, and that of Rh. antiquitatis and Eg. callabus does so most probably, i. e. in the least subsided region. Two possibilities are then to be considered: in the first place near Diessen and Esbeek more recent diluvium may have overlapped the denuded pliocene and secondly the fault postulated in the above as a straight line, may proceed more irregularly, so that in reality Esbeek and Diessen come to lie east of it. — At all events it appears from the foregoing that the young fossil mammalian remains in this part of Noord Brabant, whose geology may give us still many surprises, are rather numerous and may be of use in unravelling the tectonic of this province.

The previous discoveries near Westerhoven and Oosterhout as well as the recent ones near Esbeek and Diessen were made in superficial or nearly superficial loam deposits, which but for fossil findings, would surely be referred to the "Argiles de la Campine¹⁾". It has already previously been pointed out that these loam-deposits may be of different ages; the palaeontological findings lend support to this hypothesis.

¹⁾ J. LORÉ, Bull. Soc. Belge de Géol. XXI. 1907 p. 532—576.

Mathematics. — “*A Generalisation of MERTENS’ Theorem*”. By
M. J. BELINFANTE. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of February 24, 1923).

The theory of infinite series, which so far chiefly consisted of convergent series, being extended to the so-called summable and asymptotic series, it is natural to generalize as much as possible the classical results about convergent series to these classes of series.

For the well-known theorem of MERTENS this has been done by HARDY (BROMWICH, *Theory of Infinite Series*, p. 284), who used BOREL’s method of summation. In the present paper we treat a somewhat different generalisation, whereby we are only concerned with CESARÓ’s method of summation.

The product or the product-series of the series

$$a_1 + a_2 + \dots \quad \text{and} \quad b_1 + b_2 + \dots$$

is defined as the series $c_1 + c_2 + \dots$

where $c_i = a_1 b_i + a_2 b_{i-1} + \dots + a_i b_1$.

CESARÓ has proved: if two series are convergent, their product will be summable of order 1, and if two series are summable respectively of order p and q , their product will be summable of order $(p + q + 1)^1$.

If we call a convergent series summable of order zero, then the first part of CESARÓ’s theorem is included in the second.

MERTENS’ theorem, which runs as follows: “If one of two convergent series converges absolutely, their product is convergent” may now be stated thus:

The product of a series which is absolutely convergent, by a series which is summable of order p is summable of order zero.

In the first place we are led to the following generalisation:

Theorem 1: The product of a series which is absolutely convergent, by a series which is summable of order p , is summable of order p .

¹⁾ BROMWICH. *Theory of infinite series*, § 125 pp. 314—316.

Further we may ask for a condition, such that the product of a convergent series by a series which is summable of order p and satisfies the condition, will also be summable of order p . This can be seen from:

Theorem 2: The product of a convergent series by a series which is summable of order p and whose mean-values of order $p-1$ are limited, is summable of order p .

Finally we consider the product of two series, which are summable respectively of order p and q ; then we are led to:

Theorem 3: The product of a series which is summable of order p and whose mean-values of order $p-1$ are limited, by a series which is summable of order q , is summable of order $p+q$.

If we call a series which is summable of order p ($p \geq 1$) and whose mean-values of order $p-1$ are limited, *joinable of order p* , and if we call an absolutely converging series *joinable of order zero*, then the above theorems are included in the following:

Theorem: The product of a series which is joinable of order p , by a series which is summable of order q , is summable of order $p+q$.

The proofs however will be given separately. For the sake of completeness we begin with the deduction of some well-known formulas.

Let $x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)}, \dots$ be an arbitrary sequence of complex numbers; we define:

$$x_n^{(2)} = x_1^{(1)} + x_2^{(1)} + \dots + x_n^{(1)} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$x_n^{(k+1)} = x_1^{(k)} + x_2^{(k)} + \dots + x_n^{(k)} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We denote $x_i^{(k)}$ by $A_i^{(k)}$ if $x_i^{(1)} = 1$, whatever be i . It is easy to verify that:

$$A_n^{(k)} = \frac{(n+k-2)!}{(n-1)!(k-1)!} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We consider the series $a_1 + a_2 + \dots$ and $b_1 + b_2 + \dots$ with their product: $c_1 + c_2 + \dots$ (where $c_n = a_1 b_n + \dots + a_n b_1$), and we put:

$$\left. \begin{aligned} S_n^{(1)} &= s_n = a_1 + a_2 + \dots + a_n \quad . \\ T_n^{(1)} &= t_n = b_1 + b_2 + \dots + b_n \quad . \\ W_n^{(1)} &= w_n = c_1 + c_2 + \dots + c_n \quad . \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad (3a)$$

The quantities $S_n^{(k)}$, $T_n^{(k)}$ and $W_n^{(k)}$ are now also defined. The following identity will be satisfied for $p-i \geq 1$:

$$S_1^{(p)} T_n^{(q)} + S_2^{(p)} T_{n-1}^{(q)} + \dots + S_n^{(p)} T_1^{(q)} = S_1^{(p-i)} T_n^{(q+i)} + \dots + S_n^{(p-i)} T_1^{(q+i)} \quad (4)$$

This follows by induction:

$$\begin{aligned} & S_1^{(p)} T_n^{(q)} + S_2^{(p)} T_{n-1}^{(q)} + \dots + S_n^{(p)} T_1^{(q)} = \\ &= S_1^{(p-1)} T_n^{(q)} + (S_1^{(p-1)} + S_2^{(p-1)}) T_{n-1}^{(q)} + \dots + (S_1^{(p-1)} + \dots + S_n^{(p-1)}) T_1^{(q)} = \\ &= S_1^{(p-1)} (T_n^{(q)} + \dots + T_1^{(q)}) + S_2^{(p-1)} (T_{n-1}^{(q)} + \dots + T_1^{(q)}) + \dots + S_n^{(p-1)} T_1^{(q)} = \\ &= S_1^{(p-1)} T_n^{(q+1)} + S_2^{(p-1)} T_{n-1}^{(q+1)} + \dots + S_n^{(p-1)} T_1^{(q+1)}. \end{aligned}$$

We also prove with induction:

$$W_n^{(p+q)} = S_1^{(p)} T_n^{(q)} + S_2^{(p)} T_{n-1}^{(q)} + \dots + S_n^{(p)} T_1^{(q)} \quad (5)$$

for we have by (5):

$$\begin{aligned} W_n^{(p+q+1)} &= W_n^{(p+q)} + W_{n-1}^{(p+q)} + \dots + W_1^{(p+q)} \\ &= [S_1^{(p)} T_n^{(q)} + \dots + S_n^{(p)} T_1^{(q)}] + [S_1^{(p)} T_{n-1}^{(q)} + \dots + S_{n-1}^{(p)} T_1^{(q)}] + \dots + S_1^{(p)} T_1^{(q)} \\ &= S_1^{(p)} [T_n^{(q)} + \dots + T_1^{(q)}] + S_2^{(p)} [T_{n-1}^{(q)} + \dots + T_1^{(q)}] + \dots + S_n^{(p)} T_1^{(q)} = \\ &= S_1^{(p)} T_n^{(q+1)} + S_2^{(p)} T_{n-1}^{(q+1)} + \dots + S_n^{(p)} T_1^{(q+1)} \quad (5a) \end{aligned}$$

Finally we deduce from:

$$W_n^{(p+1)} = S_1^{(1)} T_n^{(p)} + S_2^{(1)} T_{n-1}^{(p)} + \dots + S_n^{(1)} T_1^{(p)} \quad (6)$$

with the aid of $S_n^{(1)} = a_1 + a_2 + \dots + a_n$

$$\begin{aligned} W_n^{(p+1)} &= a_1 T_n^{(p)} + (a_1 + a_2) T_{n-1}^{(p)} + \dots + (a_1 + a_2 + \dots + a_n) T_1^{(p)} \\ &= a_1 [T_n^{(p)} + T_{n-1}^{(p)} + \dots + T_1^{(p)}] + a_2 [T_{n-1}^{(p)} + \dots + T_1^{(p)}] + \dots + a_n T_1^{(p)} \end{aligned}$$

or

$$W_n^{(p+1)} = a_1 T_n^{(p+1)} + a_2 T_{n-1}^{(p+1)} + \dots + a_n T_1^{(p+1)} \quad (7)$$

The n^{th} mean-value of order p of the series $a_1 + a_2 + \dots$ respectively $b_1 + b_2 + \dots$ is defined as

$$\frac{S_n^{(p+1)}}{A_n^{(p+1)}} \quad \text{respectively} \quad \frac{T_n^{(p+1)}}{A_n^{(p+1)}}.$$

If such a mean value of order p has a limit for $n = \infty$ we call the corresponding series summable of order p^1 . By a well-known

¹⁾ In our definition the first term of a series has the index 1 and not zero as is usually the case.

theorem the summability of order p (and also the convergence) of a series implies the summability of order $(p + i)$ if $i \geq 1$).

Proof of theorem 1.

Suppose that the series $a_1 + a_2 + \dots$ is absolutely convergent (sum = s); let the series $b_1 + b_2 + \dots$ be summable of order p , or

$$\lim_{n \rightarrow \infty} \frac{T_n^{(p+1)}}{A_n^{(p+1)}} = t, \text{ then we have to prove:}$$

$$\lim_{n \rightarrow \infty} \frac{W_n^{(p+1)}}{A_n^{(p+1)}} = s \cdot t$$

Now we have:

$$W_n^{(p+1)} = a_1 T_n^{(p+1)} + a_2 T_{n-1}^{(p+1)} + \dots + a_n T_1^{(p+1)} \quad (7)$$

Put $h_n = \frac{T_n^{(p+1)}}{A_n^{(p+1)}} - t$, then $\lim_{n \rightarrow \infty} h_n = 0$. Substitution of

$$T_n^{(p+1)} = t A_n^{(p+1)} + h_n A_n^{(p+1)}$$

in (7) gives:

$$\begin{aligned} W_n^{(p+1)} &= \\ &= a_1 A_n^{(p+1)} t + a_1 A_n^{(p+1)} h_n + a_2 A_{n-1}^{(p+1)} t + a_2 A_{n-1}^{(p+1)} h_{n-1} + \dots + a_n A_1^{(p+1)} t + a_n A_1^{(p+1)} h_1 \\ &= t [a_1 A_n^{(p+1)} + a_2 A_{n-1}^{(p+1)} + \dots + a_n A_1^{(p+1)}] + \\ &\quad + [a_1 A_n^{(p+1)} h_n + a_2 A_{n-1}^{(p+1)} h_{n-1} + \dots + a_n A_1^{(p+1)} h_1]. \\ &= P + Q. \end{aligned}$$

$$\begin{aligned} P &= t [a_1 A_n^{(p+1)} + a_2 A_{n-1}^{(p+1)} + \dots + a_n A_1^{(p+1)}] = \\ &= t [s_1 A_n^{(p+1)} + (s_2 - s_1) A_{n-1}^{(p+1)} + \dots + (s_n - s_{n-1}) A_1^{(p+1)}] = \\ &= t [s_1 (A_n^{(p+1)} - A_{n-1}^{(p+1)}) + s_2 (A_{n-1}^{(p+1)} - A_{n-2}^{(p+1)}) + \dots + s_n A_1^{(p+1)}] = \\ &= t [s_1 A_n^{(p)} + s_2 A_{n-1}^{(p)} + \dots + s_n A_1^{(p)}] \\ &= t [S_1^{(1)} A_n^{(p)} + S_2^{(1)} A_{n-1}^{(p)} + \dots + S_n^{(1)} A_1^{(p)}] \\ &= t [S_1^{(p)} A_n^{(1)} + S_2^{(p)} A_{n-1}^{(1)} + \dots + S_n^{(p)} A_1^{(1)}] \\ &= t [S_1^{(p)} + S_2^{(p)} + \dots + S_n^{(p)}] \\ &= t S_n^{(p+1)}. \end{aligned}$$

$$\text{Hence } \lim_{n \rightarrow \infty} \frac{P}{A_n^{(p+1)}} = t, \lim_{n \rightarrow \infty} \frac{S_n^{(p+1)}}{A_n^{(p+1)}} = t \cdot s \text{ for } a_1 + a_2 + \dots \text{ converges}$$

absolutely and is consequently summable of order p .

¹) BROMWICH. l.c. (p. 312).

$$\begin{aligned}
Q &= a_1 A_n^{(p+1)} h_n + a_2 A_{n-1}^{(p+1)} h_{n-1} + \dots + a_n A_1^{(p+1)} h_1 \\
&= a_1 h_n [A_1^{(p)} + A_2^{(p)} + \dots + A_n^{(p)}] + a_2 h_{n-1} [A_1^{(p)} + \dots + A_{n-1}^{(p)}] + \dots + a_n h_1 A_1^{(p)} \\
&= A_1^{(p)} [a_1 h_n + a_2 h_{n-1} + \dots + a_n h_1] + A_2^{(p)} [a_1 h_n + \dots + a_{n-1} h_2] + \dots + A_n^{(p)} a_1 h_n
\end{aligned}$$

From $\lim_{n \rightarrow \infty} h_n = 0$ it is evident that M_k may be chosen, so that $|h_{k+i}| < M_k$ if $i = 1, 2, \dots$ and $\lim_{k \rightarrow \infty} M_k = 0$; putting $\sigma_n = |a_1| + |a_2| + \dots + |a_n|$ and $\lim_{n \rightarrow \infty} \sigma_n = \sigma$, we have:

$$\begin{aligned}
|Q| &< A_1^{(p)} [|a_1| \cdot |h_n| + |a_2| \cdot |h_{n-1}| + \dots + |a_n| \cdot |h_1|] + \\
&\quad + A_2^{(p)} [|a_1| \cdot |h_n| + \dots + |a_{n-1}| \cdot |h_2|] + \dots + A_n^{(p)} \cdot |a_1| \cdot |h_n| \\
&< A_1^{(p)} M_1 [|a_1| + |a_2| + \dots + |a_n|] + A_2^{(p)} M_2 [|a_1| + \dots + |a_{n-1}|] + \dots + A_n^{(p)} M_n |a_1| \\
&< A_1^{(p)} M_1 \sigma_n + A_2^{(p)} M_2 \sigma_{n-1} + \dots + A_n^{(p)} M_n \sigma_1 \\
&< \sigma [A_1^{(p)} M_1 + A_2^{(p)} M_2 + \dots + A_n^{(p)} M_n].
\end{aligned}$$

Now whatever be $\varepsilon > 0$ it is possible to calculate k so that $M_{k+i} < \frac{\varepsilon}{\sigma}$; further let M be chosen so that $M_i < M$, then we have, if $n > k$:

$$\begin{aligned}
|Q| &< \sigma [A_1^{(p)} M_1 + \dots + A_k^{(p)} M_k] + \varepsilon [A_{k+1}^{(p)} + \dots + A_n^{(p)}] \\
&< M \sigma [A_1^{(p)} + \dots + A_k^{(p)}] + \varepsilon [A_n^{(p+1)} - A_k^{(p+1)}].
\end{aligned}$$

Hence:

$$\begin{aligned}
\left| \frac{Q}{A_n^{(p+1)}} \right| &< M \sigma \frac{A_k^{(p+1)}}{A_n^{(p+1)}} + \varepsilon \cdot \frac{A_n^{(p+1)} - A_k^{(p+1)}}{A_n^{(p+1)}} \\
\lim_{n \rightarrow \infty} \cdot \frac{A_k^{(p)}}{A_n^{(p)}} &= 0 \text{ for } \lim_{n \rightarrow \infty} A_n^{(p)} = \infty \text{ because } A_n^{(p)} > A_n^{(2)} = n.
\end{aligned}$$

Hence, if n is sufficiently great:

$$\left| \frac{Q}{A_n^{(p+1)}} \right| \leq \varepsilon + \varepsilon$$

and since ε is arbitrary $\lim_{n \rightarrow \infty} \frac{Q}{A_n^{(p+1)}} = 0$.

Therefore:

$$\lim_{n \rightarrow \infty} \frac{W_n^{(p+1)}}{A_n^{(p+1)}} = \lim_{n \rightarrow \infty} \frac{P + Q}{A_n^{(p+1)}} = \lim_{n \rightarrow \infty} \frac{P}{A_n^{(p+1)}} + \lim_{n \rightarrow \infty} \frac{Q}{A_n^{(p+1)}} = s. t.$$

Proof of theorem 2.

Suppose that the series $a_1 + a_2 + \dots$ converges to s , and that the series $b_1 + b_2 + \dots$ is summable of order p (sum $= t$); further, let the mean-values of order $(p-1)$ of the series $b_1 + b_2 + \dots$ be limited.

We have to prove:

$$\lim_{n=\infty} \frac{W_n^{(p+1)}}{A_n^{(p+1)}} = s \cdot t$$

Put $S_n^{(1)} = s_n = s + h_n$, then $\lim_{n=\infty} h_n = 0$ and

$$W_n^{(p+1)} = S_1^{(1)} T_n^{(p)} + S_2^{(1)} T_{n-1}^{(p)} + \dots + S_n^{(1)} T_1^{(p)} \quad (6)$$

$$\begin{aligned} &= (s + h_1) T_n^{(p)} + (s + h_2) T_{n-1}^{(p)} + \dots + (s + h_n) T_1^{(p)} \\ &= s [T_n^{(p)} + T_{n-1}^{(p)} + \dots + T_1^{(p)}] + [h_1 T_n^{(p)} + h_2 T_{n-1}^{(p)} + \dots + h_n T_1^{(p)}] \\ &= R + S. \end{aligned}$$

$$R = s [T_n^{(p)} + T_{n-1}^{(p)} + \dots + T_1^{(p)}] = s T_n^{(p+1)}$$

$$\frac{R}{A_n^{(p+1)}} = s \cdot \frac{T_n^{(p+1)}}{A_n^{(p+1)}}$$

Since the series $b_1 + b_2 + \dots$ is summable of order p , we have:

$$\lim_{n=\infty} \frac{R}{A_n^{(p+1)}} = s \cdot \lim_{n=\infty} \frac{T_n^{(p+1)}}{A_n^{(p+1)}} = s \cdot t.$$

$$S = h_1 T_n^{(p)} + h_2 T_{n-1}^{(p)} + \dots + h_n T_1^{(p)}.$$

Since the mean-values of order $(p-1)$ are limited, it is possible to find M so that:

$$\left| \frac{T_n^{(p)}}{A_n^{(p)}} \right| < M. \text{ Hence:}$$

$$|S| < M [|h_1| A_n^{(p)} + |h_2| A_{n-1}^{(p)} + \dots + |h_n| A_1^{(p)}]. \quad (8)$$

Put $|h_n| = H_n^{(1)}$ then we have by (4)

$$\begin{aligned} |h_1| A_n^{(p)} + |h_2| A_{n-1}^{(p)} + \dots + |h_n| A_1^{(p)} &= \\ &= H_1^{(1)} A_n^{(p)} + H_2^{(1)} A_{n-1}^{(p)} + \dots + H_n^{(1)} A_1^{(p)} \\ &= H_1^{(p)} A_n^{(1)} + H_2^{(p)} A_{n-1}^{(1)} + \dots + H_n^{(p)} A_1^{(1)} \\ &= H_1^{(p)} + H_2^{(p)} + \dots + H_n^{(p)} = H_n^{(p+1)} \end{aligned}$$

The inequality (8) may now be written:

$$|S| < M \cdot H_n^{(p+1)}$$

therefore:

$$\frac{|S|}{A_n^{(p+1)}} < M \frac{H_n^{(p+1)}}{A_n^{(p+1)}}$$

Since $\lim_{n \rightarrow \infty} H_n^{(1)} = 0$ and hence $\lim_{n \rightarrow \infty} \frac{H_n^{(p+1)}}{A_n^{(p+1)}} = 0$, it follows that

$$\lim_{n \rightarrow \infty} \frac{S}{A_n^{(p+1)}} = 0.$$

Hence :

$$\lim_{n \rightarrow \infty} \frac{W_n^{(p+1)}}{A_n^{(p+1)}} = \lim_{n \rightarrow \infty} \frac{R}{A_n^{(p+1)}} + \lim_{n \rightarrow \infty} \frac{S}{A_n^{(p+1)}} = s.t.$$

Proof of theorem 3.

Suppose that the series $a_1 + a_2 + \dots$ is summable of order p , and that its mean-values of order $(p-1)$ are limited, then we have:

$$\lim_{n \rightarrow \infty} \frac{S_n^{(p+1)}}{A_n^{(p+1)}} = s \quad \text{and} \quad \left| \frac{S_n^{(p)}}{A_n^{(p)}} \right| < \text{a fixed number } M.$$

Let $b_1 + b_2 + \dots$ be summable of order q , or

$$\lim_{n \rightarrow \infty} \frac{T_n^{(q+1)}}{A_n^{(q+1)}} = t.$$

We have to prove: $\lim_{n \rightarrow \infty} \frac{W_n^{(p+q+1)}}{A_n^{(p+q+1)}} = s.t$

$$W_n^{(p+q+1)} = S_1^{(p)} T_n^{(q+1)} + S_2^{(p)} T_{n-1}^{(q+1)} + \dots + S_n^{(p)} T_1^{(q+1)} \quad (5a)$$

Put $\frac{T_n^{(q+1)}}{A_n^{(q+1)}} = t + h_n$, then $\lim_{n \rightarrow \infty} h_n = 0$

$$\begin{aligned} W_n^{(p+q+1)} &= S_1^{(p)} [t A_n^{(q+1)} + h_n A_n^{(q+1)}] + \dots + S_n^{(p)} [t A_1^{(q+1)} + h_1 A_1^{(q+1)}] \\ &= t [S_1^{(p)} A_n^{(q+1)} + \dots + S_n^{(p)} A_1^{(q+1)}] + [S_1^{(p)} h_n A_n^{(q+1)} + \dots + S_n^{(p)} h_1 A_1^{(q+1)}] \\ &= U + V. \end{aligned}$$

$$\begin{aligned} U &= t [S_1^{(p)} A_n^{(q+1)} + S_2^{(p)} A_{n-1}^{(q+1)} + \dots + S_n^{(p)} A_1^{(q+1)}] \\ &= t [S_1^{(p+q)} A_n^{(1)} + S_2^{(p+q)} A_{n-1}^{(1)} + \dots + S_n^{(p+q)} A_1^{(1)}] \\ &= t [S_1^{(p+q)} + S_2^{(p+q)} + \dots + S_n^{(p+q)}] = t . S_n^{(p+q+1)} \end{aligned}$$

$$\frac{U}{A_n^{(p+q+1)}} = t \frac{S_n^{(p+q+1)}}{A_n^{(p+q+1)}}$$

Since $\lim_{n \rightarrow \infty} \frac{S_n^{(p+1)}}{A_n^{(p+1)}} = s$, it follows that

$$\lim_{n \rightarrow \infty} \frac{U}{A_n^{(p+q+1)}} = s \cdot t$$

$$V = S_1^{(p)} h_n A_n^{(q+1)} + S_2^{(p)} h_{n-1} A_{n-1}^{(q+1)} + \dots + S_n^{(p)} h_1 A_1^{(q+1)}$$

From $\frac{|S_i^{(p)}|}{A_i^{(p)}} < M$ whatever be i , we deduce:

$$|V| < M [A_1^{(p)} A_n^{(q+1)} |h_n| + A_2^{(p)} A_{n-1}^{(q+1)} |h_{n-1}| + \dots + A_n^{(p)} A_1^{(q+1)} |h_1|]$$

hence if $1 < k < n$

$$|V| < M [A_1^{(p)} A_n^{(q+1)} |h_n| + \dots + A_{n-k}^{(p)} A_{k+1}^{(q+1)} |h_{k+1}|] + \\ + M [A_{n-k+1}^{(p)} A_k^{(q+1)} |h_k| + \dots + A_n^{(p)} A_1^{(q+1)} |h_1|].$$

Now whatever be $\varepsilon > 0$, we can calculate an integer k so that

$$|h_{k+i}| < \frac{\varepsilon}{M}; \text{ further we can find } \mu \text{ so that } |h_i| < \mu.$$

Then, if $n > k$ we have:

$$|V| < \varepsilon [A_1^{(p)} A_n^{(q+1)} + \dots + A_{n-k}^{(p)} A_{k+1}^{(q+1)}] + M\mu [A_{n-k+1}^{(p)} A_k^{(q+1)} + \dots + A_n^{(p)} A_1^{(q+1)}]$$

or, since

$$A_1^{(p)} A_n^{(q+1)} + A_2^{(p)} A_{n-1}^{(q+1)} + \dots + A_n^{(p)} A_1^{(q+1)} = A_n^{(p+q+1)} \text{ and } A_i^{(j)} \geq A_{i-1}^{(j)}$$

$$|V| < \varepsilon A_n^{(p+q+1)} + M\mu A_k^{(q+1)} [A_{n-k+1}^{(p)} + \dots + A_n^{(p)}]$$

$$< \varepsilon A_n^{(p+q+1)} + M\mu A_k^{(q+1)} A_n^{(p+1)}.$$

hence:

$$\frac{|V|}{A_n^{(p+q+1)}} < \varepsilon + M\mu A_k^{(q+1)} \cdot \frac{A_n^{(p+1)}}{A_n^{(p+q+1)}}.$$

If $q \geq 1$, $\lim_{n \rightarrow \infty} \frac{A_n^{(p+1)}}{A_n^{(p+q+1)}} = 0$ for:

$$\frac{A_n^{(p+1)}}{A_n^{(p+q+1)}} = \frac{(n+p-1)!}{(n-1)! p!} \times \frac{(n-1)! (p+q)!}{(n+p+q-1)!} \\ = \frac{(p+q)!}{p!} \times \frac{(n+p-1)!}{(n+p+q-1)!} \\ \leq \frac{(p+q)!}{p!} \times \frac{(n+p-1)!}{(n+p+1-1)!} \text{ or } \leq \frac{(p+q)!}{p!} \times \frac{1}{(n+p)}.$$

Hence we have if n is sufficiently great:

$$\frac{|V|}{A_n^{(p+q+1)}} < \varepsilon + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we have:

$$\lim_{n \rightarrow \infty} \frac{V}{A_n^{(p+q+1)}} = 0.$$

Hence:

$$\lim_{n \rightarrow \infty} \frac{W_n^{(p+q+1)}}{A_n^{(p+q+1)}} = \lim_{n \rightarrow \infty} \frac{U}{A_n^{(p+q+1)}} + \lim_{n \rightarrow \infty} \frac{V}{A_n^{(p+q+1)}} = s \cdot t$$

Remark 1.

K. KNOPP ¹⁾ ad S. CHAPMAN ²⁾ have limited the order of summability of the product of two series, which are summable of order p and q , by considering non-integral orders of summability. It may happen that the theorems proved above give more result, as is seen by the following example:

The series $1 - 1 + 1 - 1 + \dots$ is summable of order 1 and its mean-values of order 0 are limited. Hence, applying theorem 3, we see that the product of this series by a series which is summable of order p , is summable of order $(p + 1)$. Now the so-called index of summability of the series $1 - 1 + \dots$ is zero (see CHAPMAN, l. c.); the index of a series which is summable of order p , cannot exceed p : hence the index of the product cannot exceed $p + 1$, and therefore we can only infer by CHAPMAN's theory that the product is summable of order $p + 2$.

Remark 2.

HARDY ³⁾ has also given the following extension of MERTENS' theorem which is totally different from the generalisations mentioned above, and which contains MERTENS' theorem as a special case:

If Σa_n is absolutely convergent and Σb_n is a finitely oscillating series whose n^{th} term tends to zero, then their product is a finitely oscillating series, and if the limits of oscillation of Σb_n are β_1 and β_2 , those of the product are $s \cdot \beta_1$ and $s \cdot \beta_2$.

¹⁾ Sitzungsberichte der Berliner Math. Gesellschaft 1907 (p. 1—12)

²⁾ Proceedings of the London Mathematical Society, Ser. 2 Vol. 9 (p. 369—409).

³⁾ Proceedings of the London Mathematical Society, Ser. 2 Vol. 6 (p. 410—423).

Evidently the terms α_i and b_i are supposed to be real: therefore MERTENS' theorem is only a special case of this theorem when the terms of the series are real. It is however easy to see that HARDY'S proof is also valid for the following extension to series with complex terms:¹⁾

Theorem 4: If Σa_n converges absolutely to s , if $b_1 + b_2 + \dots + b_n$ is limited and $\lim_{n \rightarrow \infty} b_n = 0$, then the product of the series Σa_i and Σb_i oscillates for $n = \infty$ about the same region as the series $s \cdot \Sigma b_n$.

The functions $\varphi(n)$ and $\psi(n)$ are said to oscillate about the same region if n tends to ∞ , if the following condition is satisfied:

whatever be $\varepsilon > 0$ we can find two numbers μ and α so that it is possible to calculate whatever be $n > \mu$ a number m which satisfies the conditions:

$$|\varphi(n) - \psi(n)| < \varepsilon \quad |n - m| < \alpha$$

and that is also possible so calculate whatever be $m > \mu$ a number n which satisfies the same conditions.

Finally we prove the following theorem which is analogous to theorem 4 and which contains theorem 1 as a special case:

Theorem 5: If Σa_n converges absolutely to s , if the mean-values of order p of Σb_n are limited and the mean-values of order $(p-1)$ (which we denote by $U_n^{(p)}$) satisfy the condition:

$$\lim_{n \rightarrow \infty} \frac{U_n^{(p)}}{n} = 0,$$

then the mean-values of order p of the product-series oscillates about the same region as $s \cdot U_n^{(p+1)}$ as n tends to ∞ .

Proof of theorem 4

Substituting $p = 1$ in formula (7), we have:

$$w_n = a_1 t_n + a_2 t_{n-1} + \dots + a_n t_1$$

Hence, if $1 < k < n$:

$$w_n = [a_1 t_n + \dots + a_k t_{n-k+1}] + [a_{k+1} t_{n-k} + \dots + a_n t_1] = P + Q.$$

Suppose $|t_i| < t$ and $|s_i| < \sigma$ whatever be i .

¹⁾ It is not clear from HARDY'S article how far the author also considers series with complex terms; in the preceding pages he considers series with real terms, and his statement, as far as I am aware, is also made for real terms; yet his proof applies as well to series with complex terms.

We can find, whatever be $\varepsilon > 0$, an integer k so that:

$$|a_{k+1}| + \dots + |a_{k+p}| < \frac{\varepsilon}{3t};$$

then we have also $|s - s_k| < \frac{\varepsilon}{3t}$ and $|Q| = |a_{k+1} t_{n-k} + \dots + a_n t_1| < \frac{\varepsilon}{3}$

$$\begin{aligned} P &= a_1 t_n + \dots + a_k t_{n-k+1} = a_1 (b_1 + \dots + b_n) + \dots + a_k (b_1 + \dots + b_{n-k+1}) \\ &= (b_1 + b_2 + \dots + b_{n-k}) \cdot (a_1 + a_2 + \dots + a_k) + b_{n-k+1} s_1 + \dots + b_{n-k+1} s_k \\ &= t_{n-k} s_k + R \text{ als } R = b_n s_1 + \dots + b_{n-k+1} s_k. \end{aligned}$$

We can find μ so that $|b_{n-k+i}| < \frac{\varepsilon}{3k\sigma}$ if $n > \mu$; then we have

also $|R| < \frac{\varepsilon}{3}$ if $n > \mu$.

$$\text{Now } s_k t_{n-k} = s t_{n-k} - (s - s_k) \cdot t_{n-k}.$$

Since $|(s - s_k) \cdot t_{n-k}| < \frac{\varepsilon}{3}$ (see above), we have if $n > \mu$:

$$|P - s \cdot t_{n-k}| < \frac{2\varepsilon}{3} \quad \text{and, since } |Q| < \frac{\varepsilon}{3} \text{ and } w_n = P + Q:$$

$$|w_n - s \cdot t_{n-k}| < \varepsilon.$$

Hence we see that it is possible to calculate, whatever be $\varepsilon > 0$, an integer μ which satisfies the conditions.

Proof of theorem 5.

We have:

$$W_n^{(p+1)} = a_1 T_n^{(p+1)} + a_2 T_{n-1}^{(p+1)} + \dots + a_n T_1^{(p+1)} \quad . \quad . \quad (7)$$

Hence, if $1 < k < n$:

$$\begin{aligned} W_n^{(p+1)} &= [a_1 T_n^{(p+1)} + \dots + a_k T_{n-k+1}^{(p+1)}] + [a_{k+1} T_{n-k}^{(p+1)} + \dots + a_n T_1^{(p+1)}] \\ &= P + Q. \end{aligned}$$

$$\text{Let } \frac{|T_i^{(p+1)}|}{A_i^{(p+1)}} < t \text{ and } |s_i| < \sigma.$$

Whatever be $\varepsilon > 0$ we can find an integer k so that:

$$|a_{k+1}| + \dots + |a_{k+p}| < \frac{\varepsilon}{3t}.$$

Then $|s - s_k| < \frac{\varepsilon t}{3}$ and a fortiori:

$$\frac{|Q|}{A_n^{(p+1)}} < t \cdot [|a_{k+1}| + \dots + |a_n|] < \frac{\varepsilon}{3}$$

$$\begin{aligned} P &= [a_1 T_n^{(p+1)} + \dots + a_k T_{n-k+1}^{(p+1)}] \\ &= s_k T_{n-k}^{(p+1)} + [T_n^{(p)} s_1 + T_{n-1}^{(p)} s_2 + \dots + T_{n-k+1}^{(p)} s_k] \end{aligned}$$

Hence it follows from

$$T_{n-k}^{(p+1)} = T_n^{(p+1)} - [T_n^{(p)} + T_{n-1}^{(p)} + \dots + T_{n-k+1}^{(p)}]$$

that

$$\begin{aligned} \frac{P}{A_n^{(p+1)}} &= s_k \cdot \frac{T_n^{(p+1)}}{A_n^{(p+1)}} + \frac{T_n^{(p)} (s_1 - s_k) + T_{n-1}^{(p)} (s_2 - s_k) + \dots + T_{n-k+1}^{(p)} (s_k - s_k)}{A_n^{(p+1)}} \\ &= R + S. \end{aligned}$$

It is evident that the absolute value of S is less than

$$2\sigma \frac{|T_n^{(p)}| + |T_{n-1}^{(p)}| + \dots + |T_{n-k+1}^{(p)}|}{A_n^{(p+1)}}$$

We now prove that we can find $\mu > k$ so that if $r > \mu$:
 $\frac{|T_{r-k}^{(p)}|}{A_{r-k}^{(p+1)}} < \frac{\varepsilon}{3k\sigma}$ then it follows that $|S| < \frac{2\varepsilon}{3}$ if $n > \mu$.

For we have by hypothesis $\lim_{i \rightarrow \infty} \frac{T_i^{(p)}}{i A_i^{(p)}} = 0$.

Since $\frac{A_i^{(p+1)}}{A_i^{(p)}} = \frac{i + p - 1}{p}$ we have $\lim_{i \rightarrow \infty} \frac{T_i^{(p)}}{A_i^{(p+1)}} = 0$.

If $n > \mu$ we have:

$$\left| \frac{W_n^{(p+1)}}{A_n^{(p+1)}} - s_k \frac{T_n^{(p+1)}}{A_n^{(p+1)}} \right| < \varepsilon$$

and since $|s - s_k| < \frac{\varepsilon}{3t}$ our theorem is proved.

Remark 3.

A. ROSENBLATT (Bulletin International de l'Academie des Sciences de Cracovie", ser. A 1913 p. 612—620¹⁾) has proved the following theorem:

¹⁾ ROSENBLATT's memoir not being accessible to me, the reference above is taken from an article of G. DOETSCH, Mathematische Zeitschrift Bd. 11, p. 161—175.

If Σa_n is summable of order $p + 1$ and its mean-values of order p are limited, if Σb_n is summable of order $q + 1$ and its mean-values of order q are limited, then the product-series is summable of order $p + q + 2$.

This theorem is an extension of CAUCHY's theorem that the product of two absolutely convergent series is convergent, analogous to the extension of MERTENS' theorem given in theorem 3, and, like MERTENS' theorem implies that of CAUCHY, so theorem 3 implies that of ROSENBLATT.

Mathematics. — “On a Generalisation of TAUBER’S Theorem concerning Power Series”. By M. J. BELINFANTE. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of March 24, 1923).

Introduction.

In this paper we consider power series with complex coefficients, but for real values of the variable. We suppose them to converge if $|x| < 1$, and we denote by $x \rightarrow 1$ that x approaches 1 by real values from below.

TAUBER has proved the following theorem¹⁾:

If $\lim_{n \rightarrow \infty} na_n = 0$ and $\lim_{x \rightarrow 1} \sum a_n x^n = s$, then $\sum a_n$ converges to s .

LITTLEWOOD²⁾ has shown that the usual proof of this theorem proves more than is actually stated, and that the same proof applies to the theorem:

If $\sum_0^\infty a_n x^n$ oscillates finitely as $x \rightarrow 1$, then the limits of oscillation

as $n \rightarrow \infty$ of $\sum_0^n a_i$ are the same as the limits of oscillation of $\sum_0^\infty a_n x^n$.

In the present paper we give extensions of both theorems to the so-called mean-values of HÖLDER.

§ 1 contains the proof mentioned above and a definition of the expression “oscillate about the same region”; in § 2 the definition of HÖLDER’S mean-values and some necessary formula’s will be treated, while § 3 contains the generalizations of TAUBER’S theorem.

§ 1.

*Definition*³⁾. We say that $f(x)$ oscillates for $x \rightarrow x_0$ about the same region as $g(y)$ for $y \rightarrow y_0$, when the following conditions are satisfied:

¹⁾ Monatshefte für Math. u. Phys., 1897 Bd. 8, p. 273.

²⁾ Proc. of the Lond. Math. Soc., 1911 Vol. 9, p. 436.

³⁾ We always suppose that x resp. y approaches x_0 resp. y_0 by real values from below.

primo: it is possible whatever be $\varepsilon > 0$ and $\xi < x_0$, to calculate a number $\eta(\varepsilon, \xi)$ so that whatever be y_1 between η and y_0 , we can calculate a number x_1 between ξ and x_0 which satisfies the condition

$$|f(x_1) - g(y_1)| < \varepsilon,$$

secundo: it is possible whatever be $\varepsilon > 0$ and $\eta < y_0$ to calculate a number $\xi(\varepsilon, \eta)$ so that whatever be x_1 between ξ and x_0 we can calculate a number y_1 between η and y_0 which satisfies the condition

$$|f(x_1) - g(y_1)| < \varepsilon.$$

Theorem 1. If $\lim_{n \rightarrow \infty} na_n = 0$, $\sum_0^\infty a_n x^n$ oscillates for $x \rightarrow 1$ about the same region as $\sum_0^n a_i$ for $n \rightarrow \infty$.

Proof: We have by a well-known theorem¹⁾ that $\lim_{v \rightarrow \infty} u_v = u$ implies $\lim_{v \rightarrow \infty} \frac{1}{v} \sum_0^{v-1} u_i = u$. Hence, since $\lim_{n \rightarrow \infty} na_n = 0$ implies $\lim_{n \rightarrow \infty} n |a_n| = 0$,

$$\lim_{v \rightarrow \infty} \frac{1}{v} \sum_0^{v-1} n |a_n| = 0.$$

Therefore, whatever be $\varepsilon > 0$, we can calculate an integer μ so that if $v > \mu$:

$$v |a_v| < \frac{\varepsilon}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{1}{v} \sum_0^{v-1} n |a_n| < \frac{\varepsilon}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now, if $0 < x < 1$, we have:

$$\begin{aligned} \left| \sum_0^{v-1} a_n - \sum_0^\infty a_n x^n \right| &\leq \left| \sum_0^{v-1} a_n - \sum_0^{v-1} a_n x^n \right| + \left| \sum_v^\infty a_n x^n \right| \\ &\leq \sum_0^{v-1} a_n (1-x^n) + \left| \sum_v^\infty a_n x^n \right| \quad . \quad . \quad . \quad . \quad . \quad (3) \end{aligned}$$

$$\left| \sum_0^{v-1} a_n (1-x^n) \right| < \sum_0^{v-1} |a_n| \cdot (1-x^n) < (1-x) \cdot \sum_0^{v-1} n |a_n| \quad . \quad . \quad (4)$$

Substitution of $x = 1 - \frac{1}{v}$ in (4) gives:

¹⁾ BROMWICH, Theory of Infinite Series, p. 383.

$$\left| \sum_0^{v-1} a_n (1-x_v^n) \right| < \frac{1}{v} \sum_0^{v-1} n |a_v|$$

Hence by (2) if $v > \mu$:

$$\left| \sum_0^{v-1} a_n (1-x_v^n) \right| < \frac{\varepsilon}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Substitution of (1) in $\sum_v^\infty a_n x^n$ gives:

$$\left| \sum_v^\infty a_n x^n \right| < \sum_v^\infty \frac{\varepsilon}{2n} x^n,$$

or a fortiori:

$$< \frac{1}{v} \sum_v^\infty \frac{\varepsilon}{2} x^n$$

or

$$< \frac{\varepsilon}{2v} \cdot \frac{x^v}{1-x}$$

Substituting $x_v = 1 - \frac{1}{v}$ in the last inequality we have:

$$\left| \sum_v^\infty a_n x_v^n \right| < \frac{\varepsilon}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

From (3), (5) and (6) we deduce:

$$\left| \sum_0^{v-1} a_n - \sum_0^\infty a_n x_v^n \right| < \varepsilon$$

if $v > \mu$ and $x_v = 1 - \frac{1}{v}$, and it follows easily that both conditions of our definition are satisfied.

§ 2.

If t_1, t_2, \dots is an arbitrary sequence of quantities, we define the so-called Hölder mean-values as follows¹⁾:

$$H_n^{(1)}(t) = \frac{t_1 + t_2 + \dots + t_n}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$H_n^{(k)}(t) = \frac{H_1^{(k-1)}(t) + H_2^{(k-1)}(t) + \dots + H_n^{(k-1)}(t)}{n} \quad . \quad . \quad . \quad (8)$$

$$H_0^{(k)}(t) = H_{-1}^{(k)}(t) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (8a)$$

The following relations are easy to verify:

$$H_n^{(p)}[H^{(q)}(t)] = H_n^{(p+q)}(t) \text{ if } p \geq 1, q \geq 1. \quad . \quad . \quad . \quad . \quad (9)$$

and

¹⁾ This definition differs slightly from the usual one, as the latter is given for a series $u_1 + u_2 + \dots$ and not for a sequence.

$$n \cdot H_n^{(k)}(t) - (n-1) \cdot H_{n-1}^{(k)}(t) = H_n^{(k-1)}(t) \quad . \quad . \quad . \quad (10)$$

Let

$$s_n = a_1 + a_2 + \dots + a_n; \quad s_{-1} = s_0 = 0; \quad . \quad . \quad . \quad (11)$$

then we define:

$$H_n^{(k)}(s) = A_n^{(k)} \quad \text{if } k \geq 1, \quad n \geq -1 \quad . \quad . \quad . \quad (12)$$

$$s_n = A_n^{(0)} \quad \text{if } n \geq -1 \quad . \quad . \quad . \quad (12a)$$

$$\sigma_n^{(k)} = n \cdot [A_n^{(k)} - A_{n-1}^{(k)}] \quad \text{if } k \geq 0, \quad n > 0 \quad . \quad . \quad . \quad (13)$$

From (10), (12) and (13) we deduce:

$$\sigma_n^{(k)} = A_n^{(k-1)} - A_{n-1}^{(k)} \quad \text{if } k \geq 1 \quad . \quad . \quad . \quad (14)$$

Finally we define:

$$\varphi_k(x) = \sum_1^{\infty} [A_n^{(k)} - A_{n-1}^{(k)}] \cdot x^n \quad \text{if } k \geq 0 \quad . \quad . \quad . \quad (15)$$

thus

$$\varphi_0(x) = \sum_1^{\infty} a_n x^n \quad . \quad . \quad . \quad (15a)$$

We prove the following identities:¹⁾

$$H_n^{(1)}[\sigma^{(k)}] = \sigma_n^{(k+1)} + \frac{A_{n-1}^{(k+1)}}{n} \quad . \quad . \quad . \quad (16)$$

$$\varphi_k(x) + (1-x) \cdot \varphi'_k(x) = \frac{1}{x} \cdot \varphi_{k-1}(x) \quad . \quad . \quad . \quad (17)$$

$$(1-x) \cdot \varphi'_k(x) = \sum_0^{\infty} [\sigma_{n+1}^{(k)} - \sigma_n^{(k)}] \cdot x^n \quad . \quad . \quad . \quad (18)$$

Proof of (16).

By (14) we have:

$$\begin{aligned} \sigma_n^{(k)} + \sigma_{n-1}^{(k)} + \dots + \sigma_1^{(k)} &= \\ &= [A_n^{(k-1)} - A_{n-1}^{(k)}] + \dots + [A_2^{(k-1)} - A_1^{(k)}] + [A^{(k-1)}] \\ &= [A_1^{(k-1)} + A_2^{(k-1)} + \dots + A^{(k-1)}] - [A_1^{(k)} + A^{(k)} + \dots + A_{n-1}^{(k)}] \\ &= n \cdot A_n^{(k)} - (n-1) \cdot A_{n-1}^{(k+1)} \\ &= n \cdot A_n^{(k)} - n \cdot A_{n-1}^{(k+1)} + A_{n-1}^{(k+1)} \end{aligned}$$

hence:

¹⁾ We tacitly assume that the power series φ_k and φ_{k-1} are convergent if $-1 < x < +1$; in our applications this will be the case.

$$\begin{aligned}
 H_n^{(1)}[\sigma^{(k)}] &= \frac{\sigma_1^{(k)} + \sigma_2^{(k)} + \dots + \sigma_n^{(k)}}{n} \\
 &= A_n^{(k)} - A_{n-1}^{(k+1)} + \frac{A_{n-1}^{(k+1)}}{n} \\
 &= \sigma_n^{(k+1)} + \frac{A_{n-1}^{(k+1)}}{n}.
 \end{aligned}$$

Proof of (17).

$$\begin{aligned}
 \varphi_k(x) + (1-x) \cdot \varphi'_k(x) &= \\
 &= \sum_0^\infty x^n \left\{ A_n^{(k)} - A_{n-1}^{(k)} + (n+1) \cdot [A_{n+1}^{(k)} - A_n^{(k)}] - n \cdot [A_n^{(k)} - A_{n-1}^{(k)}] \right\} \\
 &= \sum_0^\infty x^n \left\{ [(n+1) \cdot A_{n+1}^{(k)} - n \cdot A_n^{(k)}] - [n \cdot A_n^{(k)} - (n-1) \cdot A_{n-1}^{(k)}] \right\} \\
 &= \sum_0^\infty x^n [A_{n+1}^{(k-1)} - A_n^{(k-1)}] = \frac{1}{x} \cdot \sum_0^\infty x^{n+1} \cdot [A_{n+1}^{(k-1)} - A_n^{(k-1)}] \\
 &= \frac{1}{x} \cdot \sum_1^\infty x^n \cdot [A_n^{(k-1)} - A_{n-1}^{(k-1)}] \\
 &= \frac{1}{x} \cdot \varphi_{k-1}(x).
 \end{aligned}$$

Proof of (18).

$$\begin{aligned}
 (1-x) \cdot \varphi'_k(x) &= \sum_0^\infty x^n \cdot \left\{ (n+1) \cdot [A_{n+1}^{(k)} - A_n^{(k)}] - n \cdot [A_n^{(k)} - A_{n-1}^{(k)}] \right\} \\
 &= \sum_0^\infty x^n \cdot [\sigma_{n+1}^{(k)} - \sigma_n^{(k)}].
 \end{aligned}$$

§ 3.

We prove the following extensions of TAUBER'S theorem:

Theorem 2. If $\lim_{n \rightarrow \infty} n \cdot [A_n^{(p)} - A_{n-1}^{(p)}] = 0$, and $|s_n| < c$ whatever be n , then $\sum_1^\infty a_n x^{n-1}$ oscillates as $x \rightarrow 1$ about the same region as $A_n^{(p)}$ if $n \rightarrow \infty$.

Theorem 3. If $\lim_{n \rightarrow \infty} n \cdot [A_n^{(p)} - A_{n-1}^{(p)}] = 0$ and $\lim_{x \rightarrow 1} \sum_1^\infty a_n x^n = s^1$, then we have also: $\lim_{n \rightarrow \infty} A_n^{(p-1)} = s$.

Proof of theorem 2.

From the fact that s_n is limited it is easy to deduce that $A_n^{(i)}$ is

¹⁾ See remark 2 at the end of the article.

also limited, and therefore:

$$\lim_{n \rightarrow \infty} \frac{A_{n-1}^{(k+1)}}{n} = 0.$$

Hence by (16) we conclude:

$$\lim_{n \rightarrow \infty} [H_n^{(1)}(\sigma^{(k)}) - \sigma_n^{(k+1)}] = 0 \quad . \quad . \quad . \quad . \quad (19)$$

Now it is a well-known theorem that $\lim_{n \rightarrow \infty} H_n^{(k)}(t) = s$ implies $\lim_{n \rightarrow \infty} H_n^{(k+1)}(t) = s$; ¹⁾ hence we deduce from (19) with the aid of (9):

$$\lim_{n \rightarrow \infty} [H_n^{(i+1)}(\sigma^{(k)}) - H_n^{(i)}(\sigma^{(k+1)})] = 0$$

from which we conclude:

$$\text{If } \lim_{n \rightarrow \infty} H_n^{(i)}(\sigma^{(k)}) = 0, \text{ we have also } \lim_{n \rightarrow \infty} H_n^{(i+1)}(\sigma^{(k-1)}) = 0. \quad (20)$$

By hypothesis we have: $\lim_{n \rightarrow \infty} n \cdot A_n^{(p)} - A_{n-1}^{(p)} = 0$ or by (13)

$$\lim_{n \rightarrow \infty} \sigma_n^{(p)} = 0.$$

Hence by (19)

$$\lim_{n \rightarrow \infty} H_n^{(1)}(\sigma^{(p-1)}) = 0$$

and applying (20) we get successively:

$$\lim_{n \rightarrow \infty} H_n^{(2)}(\sigma^{(p-2)}) = 0$$

$$\lim_{n \rightarrow \infty} H_n^{(3)}(\sigma^{(p-3)}) = 0$$

$$. \quad . \quad . \quad . \quad . \quad .$$

$$\lim_{n \rightarrow \infty} H_n^{(i)}(\sigma^{(p-i)}) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

HÖLDER has proved ²⁾ that if $\lim_{n \rightarrow \infty} H_n^{(i)}(t) = h$, then we have also:

$$\lim_{x \rightarrow 1} \sum_{n=0}^{\infty} (t_{n+1} - t_n) x^n = h.$$

In virtue of this theorem we have by (21):

$$\lim_{x \rightarrow 1} \sum_{n=0}^{\infty} [\sigma_{n+1}^{(p-i)} - \sigma_n^{(p-i)}] x^n = 0$$

or by (18):

$$\lim_{x \rightarrow 1} (1-x) \cdot \varphi'_{p-i}(x) = 0$$

thus by (17):

¹⁾ See BROMWICH, Theory of Infinite Series, p. 383.

²⁾ Mathematische Annalen, Bd. 20 (1882), p. 535.

$$\lim_{x \rightarrow 1} [x \cdot \varphi_{p-i}(x) - \varphi_{p-i-1}(x)] = 0.$$

Hence we infer that $\varphi_{(p-i)}(x)$ and $\varphi_{p-i-1}(x)$ oscillate about the same region as $x \rightarrow 1$. Repeating the argument for $i = 1, 2, \dots, p$, we see that

$$\varphi_p(x) = \sum_1^\infty a_n x^n \quad \text{and} \quad \varphi_p(x) = \sum_1^\infty [A_n^{(p)} - A_{n-1}^{(p)}] x^n$$

oscillate about the same region as $x \rightarrow 1$.

By hypothesis we have $\lim_{n \rightarrow \infty} n [A_n^{(p)} - A_{n-1}^{(p)}] = 0$; with the aid of theorem 1 we deduce that $\varphi_p(x) = \sum_1^\infty [A_n^{(p)} - A_{n-1}^{(p)}] x^n$ oscillates as $x \rightarrow 1$ about the same region as $A_m^{(p)} = \sum_1^m [A_n^{(p)} - A_{n-1}^{(p)}]$ if $m \rightarrow \infty$.

Combining these results we see that $\sum_1^\infty a_n x^n$ oscillates as $x \rightarrow 1$ about the same region as $A_m^{(p)}$ as $m \rightarrow \infty$.

Proof of theorem 3.

Lemma: If $\lim_{x \rightarrow 1} \varphi_{k-1}(x) = s$ and $\varphi_k(x) + (1-x) \varphi'_k(x) = \frac{1}{x} \varphi_{k-1}(x)$ then $\lim_{x \rightarrow 1} \varphi_k(x) = s$.

Proof of the lemma: If we solve the differential equation we become:

$$\varphi_k(x) = (1-x) \int_0^x \frac{\varphi_{k-1}(x)}{x(1-x)^2} dx + C(1-x)$$

Since $\lim_{x \rightarrow 1} \varphi_{k-1}(x) = s$, it is possible whatever be $\varepsilon > 0$ to calculate a number $\xi_1 < 1$ so that $\xi_1 < x < 1$ implies:

$$|\varphi_{k-1}(x) - s| < \varepsilon$$

$$\begin{aligned} (1-x) \cdot \int_0^1 \frac{\varphi_{k-1}(x)}{x(1-x)^2} dx &= (1-x) \int_0^{\xi_1} \frac{\varphi_{k-1}(x)}{x(1-x)^2} dx + (1-x) \int_{\xi_1}^x \frac{\varphi_{k-1}(x)}{x(1-x)^2} dx \\ &= (1-x) \int_0^{\xi_1} \frac{\varphi_{k-1}(x)}{x(1-x)^2} dx + (1-x) \int_{\xi_1}^x \frac{s}{x(1-x)^2} dx + (1-x) \int_{\xi_1}^x \frac{\varphi_{k-1}(x) - s}{x(1-x)^2} dx \\ &= \quad I \quad + \quad II \quad + \quad III \end{aligned}$$

Lim. $I = 0$; therefore we can calculate a number $\xi_2 > \xi_1$ so

that $|I| < \varepsilon$ if $\xi_1 < x < 1$. Further it is possible to calculate $\xi_1 > \xi_2$ so that $|II-s| < \varepsilon$ if $\xi_2 < x < 1$, for we have:

$$\begin{aligned} II &= (1-x) \int_{\xi_1}^x \frac{s \cdot dx}{x(1-x)^2} = (1-x) \cdot s \left[\log \frac{x}{1-x} + \frac{1}{1-x} - \log \frac{\xi_1}{1-\xi_1} - \frac{1}{1-\xi_1} \right] \\ &= s + \left[(1-x) \log \frac{x}{1-x} - (1-x) \log \frac{\xi_1}{1-\xi_1} - \frac{1-x}{1-\xi_1} \right] s \end{aligned}$$

and the expression between brackets tends to zero as $x \rightarrow 1$.

In like manner we can calculate $\xi > \xi_2$ so that $|III| < 2\varepsilon$ if $\xi < x < 1$. Combining these results we have if $\xi < x < 1$:

$$|I| < \varepsilon, |II-s| < \varepsilon \text{ and } |III| < 2\varepsilon,$$

therefore:

$$|I + II + III - s| < 4\varepsilon.$$

Since ε is arbitrary and $\lim_{x \rightarrow 1} C(1-x) = 0$ we infer:

$$\lim_{x \rightarrow 1} \varphi_k(x) = s.$$

We now prove theorem III as follows: by hypothesis we have

$$\lim_{x \rightarrow 1} \varphi_0(x) = \lim_{x \rightarrow 1} \sum_{n=1}^{\infty} a_n x^n = s; \text{ applying the lemma we get:}$$

$$\lim_{x \rightarrow 1} \varphi_1(x) = s; \quad \lim_{x \rightarrow 1} \varphi_2(x) = s; \quad \dots \quad \lim_{x \rightarrow 1} \varphi_p(x) = s;$$

or:

$$\lim_{x \rightarrow 1} \sum_{n=1}^{\infty} [A_n^{(p)} - A_{n-1}^{(p)}] x^n = s$$

Moreover we have by hypothesis:

$$\lim_{n \rightarrow \infty} n [A_n^{(p)} - A_{n-1}^{(p)}] = 0$$

and therefore by TAUBER'S original theorem¹⁾:

$$\lim_{n \rightarrow \infty} A_n^{(p)} = s \quad \dots \quad (22)$$

From $\lim_{n \rightarrow \infty} n [A_n^{(p)} - A_{n-1}^{(p)}] = 0$, (13) and (14) we deduce:

$$\lim_{n \rightarrow \infty} [A_n^{(p-1)} - A_{n-1}^{(p)}] = 0$$

Hence by (22)

$$\lim_{n \rightarrow \infty} A_n^{(p-1)} = s,$$

Remark 1.

It is not difficult to see that the following statement is an immediate consequence of theorem 3:

¹⁾ $\lim_{n \rightarrow \infty} n a_n = 0$ and $\lim_{x \rightarrow 1} \sum_{n=1}^{\infty} a_n x^n = s$ imply $\lim_{n \rightarrow \infty} \sum_{i=1}^n a_i = s$.

Theorem A: The conditions $\lim_{x \rightarrow 1} \sum_{n=1}^{\infty} a_n x^n = s$ and $\lim_{n \rightarrow \infty} n[A_n^{(p)} - A_{n-1}^{(p)}] = 0$ are each necessary for the existence of:

$$\lim_{n \rightarrow \infty} A_n^{(p-1)} = s,$$

and taken together they are sufficient.

Indeed the necessity of the condition $\lim_{x \rightarrow 1} \sum_{n=1}^{\infty} a_n x^n = s$ follows from HÖLDER'S theorem mentioned above, and the necessity of $\lim_{n \rightarrow \infty} n[A_n^{(p)} - A_{n-1}^{(p)}] = 0$ is seen by writing it $\lim_{n \rightarrow \infty} [A_n^{(p-1)} - A_{n-1}^{(p)}] = 0$ and by observing that $\lim_{n \rightarrow \infty} A_n^{(p-1)} = s$ implies $\lim_{n \rightarrow \infty} A_n^{(p)} = s$.

The following particular case of this theorem has been proved by TAUBER¹⁾:

Theorem B. The conditions $\lim_{x \rightarrow 1} \sum_{n=1}^{\infty} a_n x^n = s$ and

$$\lim_{n \rightarrow \infty} \frac{1}{n} (a_1 + 2a_2 + \dots + na_n) = 0$$

are both necessary for the convergence of $\sum_{n=1}^{\infty} a_n$, and taken together they are sufficient.

This may be seen by substituting $p=1$ in theorem A, for:

$$\begin{aligned} A_n^{(0)} &= s_n \\ n[A_n^{(1)} - A_{n-1}^{(1)}] &= A_n^{(0)} - A_{n-1}^{(1)} = s_n - \frac{s_1 + s_2 + \dots + s_{n-1}}{n-1} = \\ &= \frac{1}{n-1} [(n-1)s_n - (s_1 + s_2 + \dots + s_{n-1})] = \frac{1}{n-1} [(s_n - s_1) + (s_n - s_2) + \dots + (s_n - s_{n-1})] \\ &= \frac{1}{n-1} [(a_1 + a_2 + \dots + a_n) + (a_2 + \dots + a_n) + \dots + a_n] \\ &= \frac{1}{n-1} [(n-1)a_n + (n-2)a_{n-1} + \dots + a_1] \\ &= \frac{1}{n-1} \sum_{p=1}^n (p-1)a_p, \end{aligned}$$

and we may infer the equivalence of the conditions

$$\lim_{n \rightarrow \infty} \frac{1}{n-1} [a_1 + 2a_2 + \dots + (n-1)a_n] = 0 \text{ and } \lim_{n \rightarrow \infty} \frac{1}{n} (a_1 + 2a_2 + \dots + na_n) = 0$$

from the equations:

$$U(x) = a_1 x + a_2 x^2 + \dots; V(x) = a_2 x + a_3 x^2 + \dots; U(x) = a_1 x + xV(x)$$

¹⁾ BROMWICH, op. cit., p. 251.

A somewhat different generalization of theorem *B* has been given by A. KIENAST¹⁾. KIENAST defines :

$$\begin{aligned} s_n &= \sum_1^n a_k & r_n^{(1)} &= \sum_1^n k a_k \\ s_n^{(1)} &= \frac{1}{n} \sum_1^{n-1} s_k & & \\ \dots & & & \\ s_n^{(\lambda+1)} &= \frac{1}{n} \sum_{\lambda+1}^{n-1} s_k^{(\lambda)} & r_n^{(\lambda+1)} &= \sum_{\lambda}^{n-1} \frac{1}{k} r_k^{(\lambda)} \end{aligned}$$

and proves the following theorem :

Theorem C: The conditions $\lim_{n \rightarrow \infty} \frac{1}{n} r_n^{(\lambda+1)} = 0$ and $\lim_{x \rightarrow 1} \sum_1^{\infty} a_n x^n = s$ are each necessary for the existence of $\lim_{n \rightarrow \infty} s_n^{(\lambda)} = s$, and taken together they are sufficient.

The mean-values $s_n^{(\lambda)}$ differ from CESARÒ'S or HÖLDER'S mean-values, but in a second paper²⁾ KIENAST has shown the equivalence of his mean-values with those of CESARÒ-HÖLDER.

Remark 2.

We have tacitly assumed that $\sum_1^{\infty} a_n x^n$ converges if $-1 < x < 1$. This is however superfluous for our purpose as the condition $\lim_{n \rightarrow \infty} n[A_n^{(p)} - A_{n-1}^{(p)}] = 0$ implies the convergence of $\sum_1^{\infty} a_n x^n$ provided $|x| < 1$.

Indeed from $\lim_{n \rightarrow \infty} n[A_n^{(p)} - A_{n-1}^{(p)}] = 0$ we infer the absolute convergence of $\varphi_p(x) = \sum_1^{\infty} [A_n^{(p)} - A_{n-1}^{(p)}] x^n$ provided $|x| < 1$.

Further we have by (17):

$$\varphi_{k-1}(x) = x \cdot \varphi_k(x) + x(1-x) \cdot \varphi'_k(x);$$

therefore the absolute convergence of $\varphi_k(x)$, which implies the absolute convergence of $\varphi'_k(x)$, implies also the absolute convergence of $\varphi_{k-1}(x)$. Repeating the argument we infer the absolute convergence of $\varphi_0(x) = \sum_1^{\infty} a_n x^n$ provided $|x| < 1$.

¹⁾ Proceedings of the Cambridge Phil. Soc., vol. 19 (1918), p. 129.

²⁾ Proceedings of the Cambridge Phil. Soc., vol. 20 (1920), p. 74.

Chemistry. — "*Hydrogenation of Paraffin by the BERGIUS' Method*".

By Prof. H. I. WATERMAN and J. N. J. PERQUIN. (Communicated by Prof. J. BÖESEKEN).

(Communicated at the meeting of February 24, 1923).

In a previous communication on the hydrogenation by BERGIUS' method of mineral oils or allied products, different experiments were discussed, which were carried out with heavy Borneo-asphalt-oil, distillation residue (pitch) of this oil, and with asphalt obtained by distillation of Mexican crude oil¹).

The experiments in question, comprising both cracking- and hydrogenisation experiments, were executed in a vertical immovable autoclave.

That we have now chosen another material, technically perhaps of less importance for this purpose, is owing to the peculiar advantages which commercial paraffin offers for such experiments over other materials, as asphalt. Paraffin is much more easily analysed than asphalt, and this holds also for the products prepared out of paraffin, when they are compared with the corresponding substances formed in the treatment of asphalt. Thus paraffin yields products that are less strongly coloured than Mexican asphalt. For these experiments we had an autoclave at our disposal which could be shaken continuously²).

The way of procedure was for the rest quite analogous to the earlier experiments; the arrangement of the apparatus is represented in fig. 1. The capacity of the autoclave was about 2500 cm.³, the heating took place by means of gas, in such way that the temperature could be regulated accurately to a few degrees.

The paraffin had a Sp. Gr. ($15^{\circ}/15^{\circ}$) of 0,913, the solidifying point (SHUKOFF method) was $50,6^{\circ}$, the bromine-value, (addition) determined by MC. ILHINER's method³), was 0,5.

¹) Congrès international des combustibles liquides, Paris, 9—15 Octobre 1922; *Chimie et Industrie*, numéro spécial, Mai 1923, p. 200.

²) Apparatus supplied by ANDREAS HOFER, chief instrument-maker at the laboratory of Prof. Dr. FRANZ FISCHER, Kaiser Wilhelm Institut für Kohlenforschung, Mülheim—Ruhr.

³) *Journ. Am. Chem. Soc.* **16**, 275 (1894), **21**, 1084 (1899), *Journ. Soc. Chem. Ind.* **19**, 320 (1900); H. BECKURTS, *Die Methoden der Massanalyse*, Braunschweig 1913, p. 480.

Practically the bromine value of the paraffin may, therefore, be neglected. The bromine-value determined according to Mc. LHINEY's method, is obtained by subtracting the substituted bromine from the

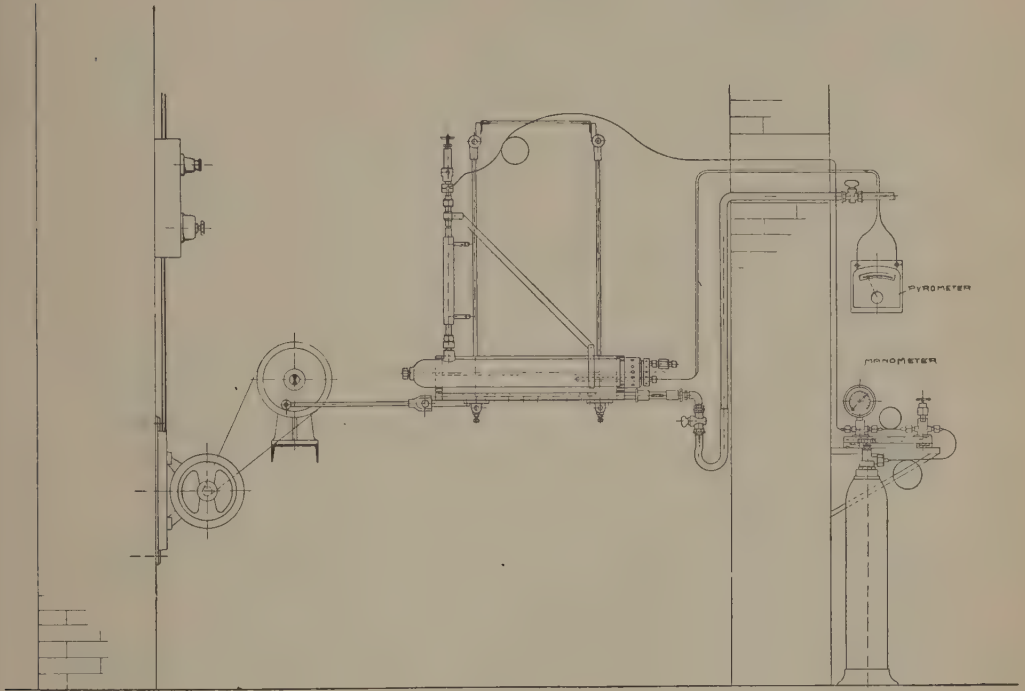
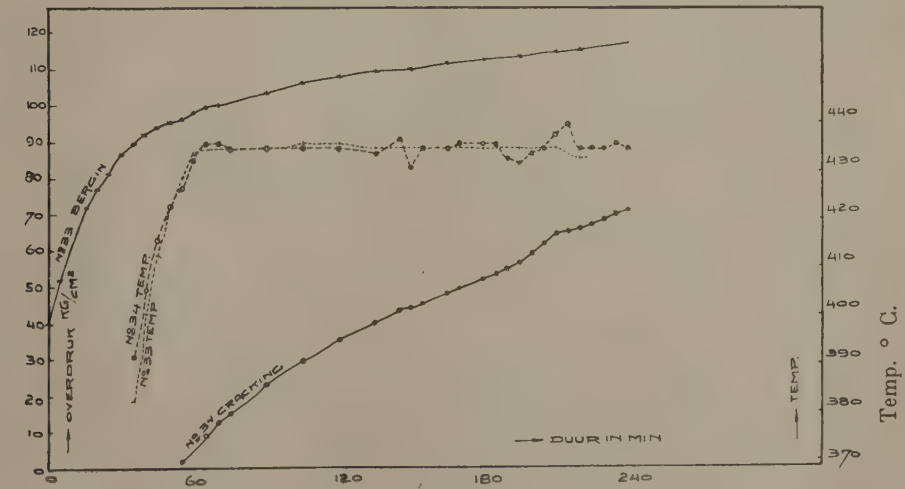


Fig. 1.



Overdruk = Pressure

Duur in min. = Time in minutes

Fig. 2.

total amount of the absorbed bromine. The remaining quantity gives a measure of the degree of unsaturation, and is expressed in percentages of weight of the weighed quantity.

In every experiment 300 gr. of paraffin was taken, an equal weight of stones being put in the autoclave to promote a thorough mixing; the temperature was always 435°. Some of the results obtained are recorded in the table, and in fig. 2 an illustration is given of the variation of the pressure in the course of experiments 33 and 34. Though in experiment 33 the typical pressure curve according to BERGIUS given in our preceding communication is not obtained, probably on account of the high temperature, the difference from the cracking-pressure curves is nevertheless very striking. In all the other experiments recorded in the table the pressure curves obtained are analogous to those of 33 and 34. The oils obtained by the BERGIUS' process were coloured from yellow to red, and perfectly transparent, a small quantity of "carbon" was deposited on the bottom. The oils obtained in cracking were very dark of colour and pretty well opaque. Here too separation of some carbon is found. The small quantity of carbon which is deposited on the bottom, when the weight of carbon which had already been deposited on the stones is added, is so small, both in the cracking and in the BERGIUS' method, that practically *the paraffin may be assumed to have been entirely converted into oil and gas in both processes.*

In this we leave out of consideration experiments 35, 37, and 40, where the duration of the processes was still so short that the reaction product had remained partially solid. Hence the product obtained had to be melted out in these experiments.

It appears from the experiments made that,

1. observations can be obtained which can be perfectly reproduced (compare 35 and 37, and 46 and 48).

2. if the duration of the experiments is long enough, the paraffin is practically quite converted into liquid oil and gas, both in the cracking and in the BERGIUS' process.

3. the yield of gasoline does not differ much in the two processes.

4. there is a great difference in the nature of the residues left in the distillation of the oil obtained according to ENGLER. Its specific gravity is always smaller in the Berginisation experiments than in the corresponding cracking experiments, which is a confirmation of corresponding experiments made by BERGIUS.

5. It appears from the final pressure, also in connection with the gas analysis (percentage of hydrogen), that actually considerable quantities of hydrogen are absorbed in the berginisation.

COMPARISON OF BERGINISATION AND CRACKING AT $\pm 435^{\circ}\text{C}$.

N ^o . of the experiment	Time required to reach the temperature in minutes	Duration of the proper experiment in minutes	Initial pressure in atmospheres	Maximum pressure during the experiment	Final pressure after complete cooling	Obtained oils in grammes	Bromine content (addition)	Distillation of the obtained oil according to ENGLER				Gases.			
								Weight % of the distilled oil.	Sp. G. residue 150/150.	Quantity in Litres	Sp. Gr. compared with air	Hydrogen percentage.			
	to 220°	to 300°	residue >300°	Loss %)											
Berginisation															
35	60	60	40	110.5	37.5	260 ¹⁾	16.4	24.9	41.0	56.3	2.7	0.846	—	0.24	85.8
37	60	60	40	108.5	37.5	272 ¹⁾	16.1	22.8	37.9	59.0	3.1	0.854	—	0.20	89.5
36	60	120	40	107	31	272	19.0	36.6	56.6	38.7	4.7	0.838	—	0.37	74.6
33	60	180	40	117	30	256	20.8	51.7	72.1	22.4	5.5	0.852	—	0.56	56.9
46	75	240	40	118	28	250	21.0	58.9	79.6	14.8	5.6	0.836	63	0.63	47.5
48	75	240	40	120	28	249	20.7	59.7	79.2	13.8	7.0	0.838	62.5	0.63	46.5
Cracking.															
40	80	60	0	30	± 2	271 ¹⁾	21.6	23.0	39.0	60.1	0.9	0.854	—	0.99	—
45	70	120	0	51.5	< 4	270	27.6	41.9	63.7	32.0	4.3	0.855	—	0.80	—
34	60	180	0	71	5	246	26.1	56.1	76.5	17.5	6.0	0.900	—	1.10	2.3
49	75	240	0	72	7.5	238	23.9	56.8	76.9	16.2	6.9	0.902	29	0.94	3.7

¹⁾ The product obtained was still solid and had to be melted out, which gave rise to extra losses of weight.

²⁾ Belongs to the lowest fraction.

6. The bromine value caused by addition of the oils obtained by berginisation is lower than that of the corresponding cracking experiments. It is, however, very risky to draw general conclusions from this bromine value, for dissolved unsaturated gases can have a great influence on the halogen value.

The example given here proves convincingly that a determination of the yield of oil and gas from a solid substance does not suffice to enable us to form a correct opinion on the process of BERGIUS. A comparative cracking experiment is required for this. Possible results refer only to the procedure followed, in this case to the periodic process, the temperature at the experimenting etc.

It is self-evident that in practice processes that proceed continuously, will be preferred. It may, however, be considered to be an established fact that when BERGIUS' method of procedure is followed, important quantities of hydrogen added from the outside, are chemically bound. After the scientific researches of SABATIER c.s. concerning the hydrogenation of hydro carbonic vapours with catalyst and the technical hardening of fatty oils (NORMANN and others), this fact, combined with the absence of express addition of catalyst, may be considered as the third great discovery in the region of hydrogenation.

*Delft, Laboratory of Chemical Technology
of the Technical-University.*

Palaeontology. — “*Contributions to our Knowledge of the Palaeontology of the Netherlands. I. Otoliths of Teleostei from the Oligocene and the Miocene of the Peel-district and of Winterswijk.*” By O. POSTHUMUS. (Communicated by Prof. J. C. SCHOUTE).

(Communicated at the meeting of February 24, 1923).

As regards the fish-fauna of the tertiary deposits in the Netherlands the occurrence has been reported of a number of Selachii in the Oligocene of South-Limburg ¹⁾, and of the Miocene of East-Gelderland ²⁾ and Overijssel ³⁾. No remains had as yet been found of Teleostei.

We are in a position to form an idea of the fish-fauna in the North Sea of Miocene time, from a number of otoliths occurring in material obtained from borings, undertaken by the Government (Institute for the Geol. Exploration of the Netherlands) on the Southern Peelhurst, notably from boring 20 (Helden) of the Middle-Miocene (75.4—80.4 m.), from boring 21 (Swalmen) of the Upper-Oligocene (100—160 m.), and of the Middle-Miocene (75—100 m.); likewise in material originating from boring 22 (Liessel) also of Middle-Miocene date (100—190 m.).

Moreover the test-boring U near Winterswijk, placed at my disposition some otoliths from the Septarian clay, and from the Middle-Miocene, laid bare in the bed of Slingerbeek near Winterswijk.

The following specimens have been found ⁴⁾:

Oligocene.

Middle-Oligocene (Septaria clay), Winterswijk.

Otolithus (Scopelus) pulcher, Prochazka.

¹⁾ W. C. H. STARING. De bodem van Nederland, 2e deel, Haarlem, 1860, p. 282.

²⁾ Ibid, p. 209, 210.

³⁾ T. C. WINKLER. Catalogue systématique du Musée Teyler, 6me livr. 1867, p. 624.

⁴⁾ They will before long be figured and described in a more detailed memoir.

Upper-Oligocene, Swalmen.

- Otolithus (Dentex) nobilis, Koken.
 „ (Percidarum) limburgensis, nov. spec.
 „ (Trachinus) mutabilis, Koken.
 „ (Trigla) Schuberti, nov. spec.
 „ (Scopelus) austriacus, Koken.
 „ (Scopelus) pulcher, Prochazka.
 „ (Gonostoma?) parvulus, Koken.
 „ (Gonostoma?) angustus, nov. spec.
 „ (Fierasfer) nuntius, Koken.
 „ (Gadus) elegans, Koken.
 „ (Merlangus) cognatus, Koken.

Miocene.

Middle-Miocene, Swalmen.

- Otolithus (Percidarum) frequens, Koken.
 „ (Trachinus) mutabilis, Koken.
 „ (Trigla) rhombicus, Schubert.
 „ (Gobius) aff. elegans, Prochazka.
 „ (Ophidiidarum) semiglobosus, nov. spec.
 „ (Ophidiidarum) swalmensis, nov. spec.
 „ (Gonostoma?) parvulus, Koken.
 „ (Solea) approximatus, Koken.
 „ (Rhombus) rhenanus, Koken.
 „ (incertae sedis) peelensis, nov. spec.

Middle-Miocene, Helden.

- Otolithus (Serranus) Noetlingi, Koken.
 „ (Centropristis) integer, Schubert.
 „ (Dentex) nobilis, Koken.
 „ (Percidarum) acuminatus, nov. spec.
 „ (Trigla) Schuberti, nov. spec.
 „ (Sciaenidarum) Staringi, nov. spec.
 „ (Gonostoma) aff. gracilis, Prochazka.
 „ (Clupea) testis, Koken.
 „ (Clupea) Priemi, nov. spec.
 „ (Gadus) elegans, Koken.
 „ (Phycis) elongatus, nov. spec.
 „ (incertae sedis) Mariae, Schubert.
 „ (incertae sedis) peelensis, nov. spec.

Middle-Miocene, Liessel.

- Otolithus (*Dentex*) *nobilis*, Koken.
 „ (*Percidarum*) *frequens*, Koken.
 „ (*Percidarum*) *Liesselensis*, nov. spec.
 „ (*Scopelus*) *austriacus*, Koken.
 „ (*Scopelus*) *pulcher*, Prochazka.
 „ (*Gonostoma*?) *parvulus*, Koken.
 „ (*Clupea*) *testis*, Koken.
 „ (*Fierasfer*) *nuntius*, Koken.
 „ (*Gadus*) *elegans*, Koken.
 „ (*Merluccius*) *emarginatus*, Koken.
 „ (*Phycis*) *elongatus*, nov. spec.
 „ (*Hymenocephalus*) *globosus*, nov. spec.
 „ (*Hymenocephalus*) *medius*, nov. spec.
 „ (*Hymenocephalus*) *ovalis*, nov. spec.
 „ (*Hymenocephalus*) *Brinki*, nov. spec.
 „ (*Hymenocephalus*) *dubius*, nov. spec.
 „ (*Macrurus*) *pusillus*, nov. spec.
 „ (*Macrurus*) *ellipticus*, Schubert.
 „ (*Macrurus*) *debilis*, nov. spec.

Middle-Miocene, Winterswijk.

Otolithus (*Gadus*) *elegans*, Koken.

The fauna of the Upper-Oligocene of Swalmen is characterised by the absence of littoral forms; the fishes that occur, inhabit the deeper and more open parts of the sea, as e.g. *Dentex*, especially in the upper water-layers, or the *Scopelidae*, especially at greater depth. The depth may have been somewhere about 400 m. at a moderate distance from the shore. This tallies with the known data, as the Upper-Oligocene is represented in erosion-rests as far as the line Liege—Aachen—Cologne.

From Middle-Miocene data are known from localities on the Southern Peelhurst, lying in one line, that is about straight and runs about S.E.—N.W. In the South-most of these three localities, near Swalmen, the genera *Rhombus*, *Solea* and *Gobius* are conspicuous. They are all littoral forms, and not met with in the material of Helden, about 20 km. farther, where, however, *Clupea*, *Serranus*, and *Dentex* occur; these fishes we also find near Liessel, about 18 km. farther in Noord-Brabant, where, however, *Macruridae* and *Scopelidae* predominate in the material. Judging from the remains of fishes Swalmen is not far from the ancient coastline; in the vicinity of

Heldén the fauna resembles closely that of a moderately deep sea, while the remains of Macruridae, occurring in the material of Liessen, originate from deep-sea forms, so that here we have to assume a greater depth of about 1000 m. This conclusion is in accordance with the results of the inquiries of the Government Institute for the Geological Exploration of the Netherlands: the boundary-line between the continental and the marine Miocene runs about via Swalmen; the lignite formation occurs near Melick-Herkenbosch and Vlodrop, while in the profile of boring 21 the lowermost layers of the Miocene are marine, and the upper layers display a limnic facies. It seems to me that a closer inspection of material from the Groote Slenk, southwest of the Peelhurst, would be very interesting.

The tertiary fauna of this region differs from the recent fauna of the North-Sea: on the one side forms occur that inhabit greater depths than those living in the North Sea at the present day, such as Scopelidae and Macruridae, which occasionally occur at high latitudes in the Atlantic Ocean; on the other side the tertiary fauna comprises genera such as *Dentex*, *Centropristis* and *Serranus*, now living at lower latitudes. In my judgment the occurrence of the latter points to a change of environment, which is to be ascribed either directly to a change of climate, or to other conditions, e.g. an altered direction of the oceanic currents.

In conclusion I wish to express my warm thanks to Prof. Dr. J. H. BONNEMA for kindly placing at my disposal the material in the Geological-Mineralogical Institute of the State University of Groningen.

Palaeontology. — “Contributions to our Knowledge of the Palaeontology of the Netherlands”. II. “On the Fauna of the Phosphatic Deposits in Twente. (Lower Oligocene)” By O. POSTHUMUS. (Communicated by Prof. J. F. VAN BEMMELÉN).

(Communicated at the meeting of March 24, 1923).

In examining a collection of fossils, derived from the phosphatic-nodulus-bearing deposits of the localities Ootmarsum and Rossum (between Oldenzaal and Denekamp) I came upon the following formations:

Coeloma balticum SCHLÜTER, Zeitschrift der deutschen Geol. Ges. Bd. 31, 1879, p. 604, Pl. XVIII; one specimen.

Myliobates toliapicus L. AGASSIZ, Recherches sur des Poissons fossiles, vol. 3, 1843, p. 321, tab. 47, fig. 15—20; loose toothplates.

Carcharodon angustidens L. AGASSIZ, Recherches etc., vol. 3, 1843, p. 255, tab. fig. 20—25, tab. 30, fig. 3; teeth.

Notidanus primigenius L. AGASSIZ, Recherches etc., vol. 3, 1843, p. 218, tab. 27, fig. 4—8, 13—17; teeth.

Oxyrhina Desori (L. AGASSIZ) SISMONDA, Memoria della Reale Accademia delle Scienze di Torino, 2d series, t. X, 1849, p. 44, tab. II, fig. 7—16; teeth.

Oxyrhina Desori L. SISMONDA mut. flandrica, M. LERICHE, Mémoires du Musée Royal d'histoire naturelle de Belgique, T. 5, p. 280, fig. 87; vertebrae.

Odontaspis cuspidata L. AGASSIZ, Recherches etc., vol. 3, 1843, p. 294, tab. 37, fig. 43—49; teeth.

Otodus obliquus L. AGASSIZ, Recherches etc., vol. 3, 1843, p. 267, tab. 31, tab. 36, fig. 22—27; teeth.

Lamna spec., vertebrae.

Phyllodus polyodus L. AGASSIZ, Recherches etc., vol. 2, 1843, p. 240, tab. 69a, fig. 6, 7;

And in addition some fragments of bone, presumably from Cetacea.

The phosphatic deposits are disposed in the profile as follows¹⁾:

“Underlying the Middle-Oligocene Septarian clay are pale-green, very fine glauconite sands, probably referable to Lower-Oligocene, but seeming to belong to the Middle-Oligocene. At the basis of these sands a very typical conglomerate layer of loosened phosphorite nodules and shark's teeth appears, as may be found e.g. in the eocene quarries at the southern base of Lonnekerberg in the neighbourhood of Rossum, between Oldenzaal and Denekamp, and in the hills north of Ootmarsum”. The phosphatic deposits

¹⁾ Eindverslag van de Rijksopsporing van Delfstoffen. Amsterdam, 1918, p. 114.

therefore may be estimated to be of Lower-Oligocene date; at all events they must have been formed at the commencement of the Oligocene transgression.

These formations are best compared with the Oligocene phosphatic deposits of the North-German Plain, of which those from Helmstedt have become familiar to us through the researches of VON KOENEN and H. B. GEINITZ¹⁾. It appears that all the fossils found in Twente, except *Oxyrphina Desori*, are also to be found near Helmstedt, which proves that the two deposits are equivalent.

This induces me to put forward some remarks about the formation of phosphatic nodules. Most authors advocate the view that the more or less rounded shape of these bodies is to be attributed to transportation, which view is adhered to by recent observers, as shown by the "Eindrappot" from which we just now quoted a passage. We contend that the nodules, in many cases, are not rounded, but more or less irregular, nay, as STARING²⁾ observes, they often seem to be made up of two or more rounded nodules. The shark's teeth are in many cases enclosed in an approximately rounded phosphatic nodule: the portion that is sticking out, however, is not worn off at all, which fact clashes with the presumable genesis. H. B. GEINITZ assumed the transport of the nodules to have taken place in the Recent Tertiary and based this view on the fact of their presence in the layers of *Myliobates* and of *Lamna cuspidata*, which he had examined, and which up to that time had been recognized only in the Pliocene. Now, this cannot apply to the Overijssel phosphatic deposits, in which these remains have also been met with, because the younger deposits of the Oligocene also occur here. The palaeontological argument that the rounded shape is attributable to rolling cannot be sustained. We are bound to assume that after the formations of the phosphate-concretions, the position of the deposits remained unaltered, which conception has been supported already by Dr. W. P. A. JONKER³⁾ on other grounds.

I wish to conclude by gratefully acknowledging my indebtedness to Mr. J. BERNINK, Director of the Museum "Natura Docet" at Denekamp, for granting me access to the fossils collected by him.

¹⁾ H. B. GEINITZ, Die sogenannten Koproliethenlager von Helmstedt, Büddenstedt und Schleweke bei Harzburg. Abhandlungen der Naturwiss. Gesellschaft „Isis" in Dresden. 1883, p. 3—14.

H. B. GEINITZ, Ueber neue Funde in den Phosphatlagern von Helmstedt, Büddenstedt und Schleweke. Isis, 1883, p. 37—46.

²⁾ W. H. C. STARING, De bodem van Nederland. 2e deel. Haarlem, 1860, p. 195.

³⁾ W. P. A. JONKER, Het ontstaan van phosphorieten. Handelingen van het 17e Natuur- en Geneeskundig Congres, 1920, p. 94—96.

Mathematics. — “An application of the theory of integral equations on the determination of the elastic curve of a beam, elastically supported on its whole length”. By Prof. C. B. BIEZENO. (Communicated by Prof. J. C. KLUIJVER).

(Communicated at the meeting of March 24, 1923).

In his well-known treatise „Vorlesungen über Technische Mechanik” (Vol. III, § 48) FÖPPL describes a construction, by which the elastic curve of a beam, elastically supported on its whole length, might be approximated.

If in the differential equation of this elastic curve

$$EI y'''' + ky = q$$

(EI = coefficient of stiffness of the beam, k = coefficient of stiffness of the supporting ground, q = specific continuous loading) the function y where known, it would be possible to refine this function by integrating four times the expression $\frac{q - ky}{EI}$.

This integration would graphically correspond to the construction of the elastic curve of a beam, which carries only well-known forces.

It is obvious, therefore, first to make a supposition about the elastic curve — in such a way, of course, that the reaction-forces of the supporting ground will be in equilibrium with the external forces of the beam —, then to integrate graphically the expression $\frac{q - ky}{EI}$, and finally to control, if the before-mentioned accordance takes place.

„Im allgemeinen — such is the opinion of FÖPPL — wird man zunächst eine starke Abweichung in der Gestalt beider Kurven finden. Dann ändert man die zuerst gezeichnete Belastungsfläche so ab, dass sich die Lastverteilung jetzt der Gestalt der gefundenen elastischen Linie nähert und wiederholt das Verfahren für diese zweite Annahme. Die Uebereinstimmung zwischen Belastungsfläche und zugehöriger elastischen Linie wird jetzt besser werden und nach mehrmaliger Wiederholung findet man mit hinreichender Genauigkeit die wirkliche Druckverteilung.”

Certainly it will be possible, — under favourable conditions — to find in this way technical sufficient accordance between the supposed curve and the one, derivated from it; but generally the convergency of the described process is uncertain.

In the following paper a convergent process will be given.

2. The equation

$$E I y'''' + k y = q$$

is transformed in

$$y'''' + k' y = q'$$

if $\frac{k}{EI} = k'$, $\frac{q}{EI} = q'$.

Putting $y'''' = \varphi(x)$ it becomes:

$$\varphi(x) + k' \int_0^x \varphi(x) dx = q' + Ax^3 + Bx^2 + Cx + D$$

or, using the well-known relation

$$\int_0^x \varphi(x) dx = \int_0^x \frac{(x-s)^3}{3!} \varphi(s) ds$$

$$\varphi(x) + k' \int_0^x \frac{(x-s)^3}{3!} \varphi(s) ds = q' + Ax^3 + Bx^2 + Cx + D.$$

A , B , C and D are constants of integration, which enable us to satisfy the following conditions:

1°. $y'' = 0$, $y''' = 0$ at $x = 0$.

2°. $y'' = 0$, $y''' = 0$ at $x = l$.

The former conditions imply, as is seen from the relation

$$y = \int_0^x \varphi(x) dx = \frac{Ax^3 + Bx^2 + Cx + D}{k'}$$

that the coefficients A and B are zero. The coefficients C and D are determinated by the latter conditions.

3. According to VOLTERRA the solution of the integralequation

$$\varphi(x) + k' \int_0^x \frac{(x-s)^3}{3!} \varphi(s) ds = q' + Cx + D$$

may be written as:

$$\varphi(x) = \varphi_0(x) + k^1 \varphi_1(x) + k^2 \varphi_2(x) + k^3 \varphi_3(x) + \dots$$

where

$$\begin{aligned}\varphi_0(x) &= q' + Cx + D \\ \varphi_1(x) &= - \int_0^x \frac{(x-s)^2}{3!} \varphi_0(s) ds \\ \varphi_2(x) &= - \int_0^x \frac{(x-s)^2}{3!} \varphi_1(s) ds \\ &\vdots \\ \varphi_n(x) &= - \int_0^x \frac{(x-s)^2}{3!} \varphi_{n-1}(s) ds.\end{aligned}$$

This solution however can only graphically be used, if the coefficients C and D are known. Nevertheless these coefficients depend on the second and first integral of $\varphi(x)$ in a point which is different from zero. Therefore we cannot find them a priori.

4. To meet this difficulty, we introduce the function

$$\chi_0(x) = q' + C_0 x + D_0;$$

C_0 and D_0 being two constants, determined by:

$$\begin{aligned}\int_0^l \chi_0(x) dx &= 0 \\ \int_0^l \chi_0(x) \cdot x dx &= 0.\end{aligned}$$

By choosing C_0 and D_0 in this manner, we reach that 1°. C_0 and D_0 can easily be graphically found, and 2°. that the function

$$\overline{\varphi}_1(x) = - \int_0^x \frac{(x-s)^2}{3!} \chi_0(s) ds$$

satisfies at the point $x=l$ the conditions

$$\overline{\varphi}_1''' = 0, \quad \overline{\varphi}_1'' = 0,$$

or the conditions

$$\int_0^l \chi_0(x) dx = 0, \quad \int_0^l dx \int_0^x \chi_0(x) dx = 0$$

For:

$$\overline{\varphi''}_1(x)_{x=l} = - \int_0^l dx \int_0^x \chi_0(x) dx = \left\{ -x \int_0^x \chi_0(x) dx \right\}_0^l + \int_0^l x \chi_0(x) dx = 0.$$

If we should deduce the function $\overline{\varphi}_2(x)$ from $\overline{\varphi}_1(x)$, in the manner which VOLTERRA indicates, the second and third derivatives of $\overline{\varphi}_2(x)$ would not be zero at the point $x=l$. Therefore we define the function

$$\chi_1(x) = - \left[\int_0^x \frac{(x-s)^2}{3!} \chi_0(s) ds + C_1 x + D_1 \right]$$

C_1 and D_1 being constants determinated by

$$\begin{aligned} \int_0^l \chi_1(x) dx &= 0 \\ \int_0^l \chi_1(x) \cdot x dx &= 0. \end{aligned}$$

In this way, the second and third derivatives of $\chi_1(x)$ take at the points $x=0$ and $x=l$ the prescribed values; on the other hand fore-fold integration of $\chi_1(x)$ gives rise to a function, the second and third derivatives of which are at the point $x=l$ also equal to zero.

This being stated, we are lead to define the series of functions

$$\begin{aligned} \chi_0(x) &= q' + C_0 x + D_0 \\ \chi_1(x) &= - \left[\int_0^x \frac{(x-s)^2}{3!} \chi_0(s) ds + C_1 x + D_1 \right] \\ \chi_2(x) &= - \left[\int_0^x \frac{(x-s)^3}{3!} \chi_1(s) ds + C_2 x + D_2 \right] \\ &\vdots \\ \chi_n(x) &= - \left[\int_0^x \frac{(x-s)^n}{3!} \chi_{n-1}(s) ds + C_n x + D_n \right] \end{aligned}$$

where the coefficients C_i and D_i are bound by the conditions

$$\begin{aligned} \int_0^l \chi_i(x) dx &= 0 \\ \int_0^l \chi_i(x) \cdot x dx &= 0 \end{aligned}$$

and to put

$$\varphi = \chi_0(x) + k' \chi_1(x) + k'^2 \chi_2(x) + \dots$$

This function satisfies formally the equation

$$\varphi(x) + k' \int_0^x \frac{(x-s)^2}{3!} \varphi(s) ds = q' + Cx + D$$

and the expression y , which follows from it:

$$\begin{aligned} y &= \frac{q' - \varphi}{k'} = \frac{q' - (q' + C_0 x + D_0) - k' \chi_1(x) - k'^2 \chi_2(x) - \dots}{k'} = \\ &= -\frac{C_0 x + D_0}{k'} - \chi_1(x) - k' \chi_2(x) - k'^2 \chi_3(x) \dots \end{aligned}$$

satisfies formally the conditions, imposed at the ends $x = 0$ and $x = l$.

For, substituting the expression φ in the integral equation we obtain — provided that it be allowed to integrate term by term the series, which occurs under the sign of integration:

$$C_0 x + D_0 - k' (C_1 x + D_1) - k'^2 (C_2 x + D_2) - \dots = Cx + D.$$

If the series, which appears in the first member of this equation, converges, there can be disposed of the constants C and D in such a manner, that the equation becomes an identity.

Of course it would now be necessary to examine the convergency of the described process of iteration.

For this investigation however we refer to the paper of Mr. J. DROSTE, which follows this. We will state here only, that convergency is sure, if $\frac{k'l^3}{EI} < 500$, and go on to demonstrate in which manner the process can be graphically performed.

5. At the first place the system of forces, which loads the beam, is substituted by another load, changing linearly, ($q_0 = \alpha x + \beta$), and which is statically equivalent with the first.

This substitute load causes a sinking down of the beam, determined by

$$y_0 = \frac{\alpha x + \beta}{k}.$$

This y_0 can be considered as the first approximation of the required y , and can be brought in relation with the expression $C_0 x + D_0$, which is defined in N°. 3.

Indeed, $\alpha x + \beta$ satisfies the equations

$$\int_0^l (\alpha x + \beta) dx = \int_0^l q dx$$

$$\int_0^l (\alpha x + \beta) \cdot x dx = \int_0^l q \cdot x dx$$

on the contrary $C_0 x + D_0$ is defined by

$$\int_0^l (C_0 x + D_0) dx = - \int_0^l q' dx = - \int_0^l \frac{q}{EI} dx$$

$$\int_0^l (C_0 x + D_0) \cdot x dx = - \int_0^l q' \cdot x dx = - \int_0^l \frac{q}{EI} \cdot x dx.$$

It follows, that $\alpha x + \beta \equiv -EI(C_0 x + D_0)$, so that:

$$y_0 = \frac{\alpha x + \beta}{k} = - \frac{C_0 x + D_0}{k'}.$$

The load which really charges the beam differs from the substitute load by:

$$q_1 = q - q_0 = q - (\alpha x + \beta) = EI(q' + C_0 x + D_0) = EI\chi_0(x).$$

By adding this load (which is in equilibrium) to the load q_0 , we would regain the real conditions of loading.

However, if we add the load q , the beam gets a deflexion y_1 , determinated by:

$$EI y_1''' = EI \chi_0(x)$$

Hence:

$$y_1 = \int_0^x \chi_0(x) dx = \int_0^x \frac{(x-s)^2}{3!} \chi_0(s) ds + A_1 x^3 + B_1 x^2 + C_1 x + D_1.$$

The second and third derivates of y_1 being zero for $x=0$, it follows that $A_1=0$, $B_1=0$.

Choosing C_1 and D_1 so that:

$$\int_0^l y_1 dx = 0$$

$$\int_0^l y_1 \cdot x dx = 0.$$

we identify y_1 and $-\chi_1(x)$.

At the same time, the forces, defined by ky_1 , are in equilibrium.

If the elastic ground were loaded with ky_1 , it would obtain the deflexion y_1 . In this case the beam and the ground would have the same shape. However the load on the ground can only arise from the beam. The deflexion y on the ground therefore involves necessarily a reaction-load $-ky_1$ on the beam.

This latter load gives rise to another deflexion y_2 of the beam, defined by:

$$EI y_2''' = -ky_1 = k\chi_1(x)$$

Hence

$$y_2 = k' \left\{ \int_0^x \frac{(x-s)^2}{3!} \chi_1(s) ds + C_2 x + D_2 \right\}.$$

If we require again that the load ky_2 , which follows from y_2 , is in equilibrium, we find that:

$$y_2 = -k' \chi_2(x).$$

From this, we deduce $y_3 = -k'^2 \chi_3(x)$ and so on. Therefore, the terms of the series:

$$y = -\frac{C_0 x + D_0}{k'} - \chi_1(x) - k' \chi_2(x) - k'^2 \chi_3(x) \dots$$

represent elastic curves of a beam, which is loaded in a well-defined manner.

6. Fig. 1 illustrates the described construction in the case: $l = 200$ cm., $b =$ breadth of the beam $= 25$ cm., $I = 5000$ cm⁴, $E = 100000$ kg/cm²; $EI = 5 \times 10^8$ kg.cm², $\bar{k} = 5$ kg/cm², $k = b\bar{k} = 125$ kg/cm². The load diagram has a parabolic form; the specific load at the ends of the beam is $1/4$ of its value at the middle. The total load is 15000 kg. The scale of length in horizontal direction is $n = 5$ (1 cm \longleftrightarrow means 5 cm \longleftrightarrow).

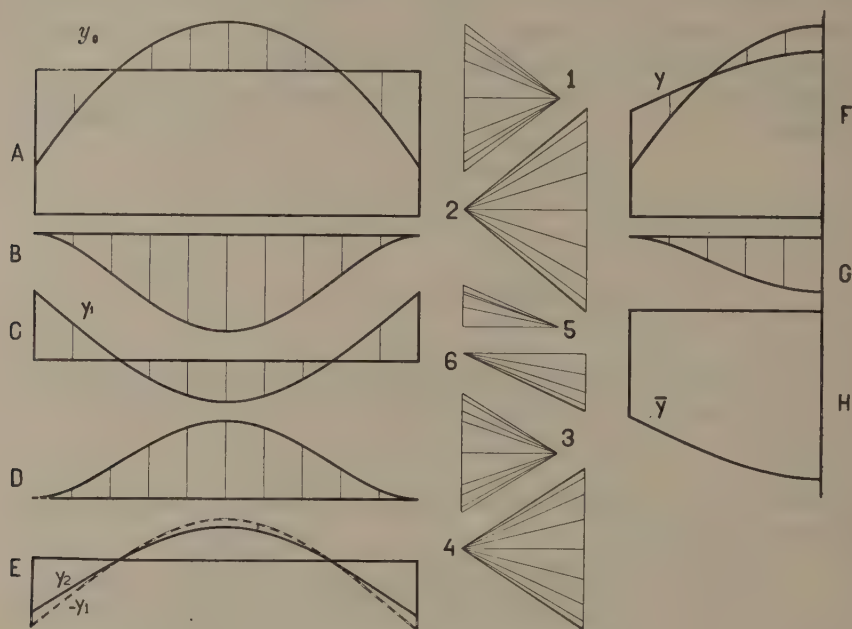
The deflexion are 25 times enlarged; 1 cm. \updownarrow represents $1/25$ cm \updownarrow .

The linear load q_0 , which is statically equivalent to the given load q , will give a sinking down to the beam, which is:

$$y_0 = \frac{15000 \text{ KG}}{125 \text{ KG/cm}^2 \times 200 \text{ cm}} = 0,6 \text{ cm.}$$

This sinking down is represented in figure 1a by $25 \times 0,6$ cm. $= 15$ cm.; and gives rise to the straight line y_0 . This line also

represents, when the scale is altered, the load q_0 ; in this case 1 cm. \updownarrow must be interpreted as $\frac{15000 \text{ kg.}}{200 \times 15 \text{ cm}} = 5 \text{ kg/cm}$ (say $m_1 \text{ kg/cm}$).



On this scale the parabolic load q has been drawn in fig. 1a, so that the load $q - q_0$, — which determines the elastic curve y_1 — is represented in fig. 1a by the hatched area.

In the well-known manner the elastic curve y_1 , which corresponds to the load $q - q_0$, is constructed (see figures 1b and 1c with the corresponding pole figures 1 and 2).

To determine the situation of the pole in the second pole figure, we make the following remarks.

In figure 1a 1 cm. \longleftrightarrow represents n cm. \longleftrightarrow ; 1 cm. \updownarrow represents $m_1 \text{ kg/cm}$. Therefore 1 cm² of fig. 1a represents $nm_1 \text{ kg}$.

Assuming now that in the first pole figure 1 cm. (whether \longleftrightarrow or \updownarrow) will represent $m_2 \text{ cm}^2$ of figure 1a (in the drawing m_2 is supposed to be 5) and that the first pole distance has a length of $H_1 \text{ cm}$ (in the drawing 10 cm), we see that H_1 represents $m_1 m_2 n H_1 \text{ kg}$.

Hence 1 cm. \updownarrow in fig. 1b represents $m_1 m_2 n^2 H_1 \text{ kg. cm}$. Consequently

the unity of area in fig. 1b means in the next integration $\frac{m_1 m_2 n^2 H_1}{EI}$ units.

The second pole distance H_2 therefore represents $\frac{m_1 m_2 m_3 n^4 H_1 H_2}{EI}$ units, if we suppose that 1 cm. of this distance represents m_3 cm² (in the drawing 10 cm²) of the area in fig. 1b.

From all this it follows finally that 1 cm. \updownarrow in fig. 1c represents

$$\frac{m_1 m_2 m_3 n^4 H_1 H_2}{EI} \text{ cm.}$$

Now the elastic curves y_1 and y_0 must be drawn on the same scale; hence:

$$\frac{m_1 m_2 m_3 n^4 H_1 H_2}{EI} = 1/25,$$

$$H_2 = \frac{1}{25} \frac{EI}{m_1 m_2 m_3 n^4 H_1} = 12.8 \text{ cm.}$$

The elastic curve y_1 once found, the drawing process is to be repeated so many times, that the last approximations may be neglected. By adding the different curves y_0, y_1, y_2, \dots we obtain the elastic curve y . The final result can be controlled as follows. We load the beam at the one side by the well-known external forces, at the other side by the continuous load ky , which follows from the elastic curve y . Then we construct the elastic curve \bar{y} . If the result y were exact, the curves y and \bar{y} must be identical. Fig. 1f, g, h shows, that a difference between the curves y and \bar{y} cannot be observed.

7. Considering fig. 1, it appears that the ordinates of the curves y_2 and y_1 are proportional. If the factor of proportionality is called $-\mu$, so that $y_2 = -\mu y_1$, it is easily seen that the ordinates of the curve y_3 can be written as $-\mu y_2$, and so on.

The ordinates y_1, y_2, \dots, y_n at any point can therefore be looked upon as terms of a geometrical series and the curve y can be obtained by adding y_0 to the sum of all the following approximations.

Not only when the factor of proportionality μ is < 1 , but also when $\mu > 1$, it may occur that the described drawing process is useful to find the elastic curve.

Supposing that the load $-ky_n$ gives rise to the deflexion $-\mu y_n$ there can be found a factor v , such that the function $v y_n$

satisfies the equation $EI y'''' + ky = -ky_n$. Using the relation $-EI \mu y_n'''' = -ky_n$, we find the condition:

$$v \frac{ky_n}{\mu} + kv y_n = -ky_n$$

whence:

$$v = \frac{-\mu}{\mu + 1}.$$

We therefore can obtain the deflexion y of the beam by adding $\frac{-\mu}{\mu + 1} y_n$ to the sum of the curves $y_0, y_1 \dots y_n$, or by adding

$$\left(1 + \frac{-\mu}{1 + \mu}\right) y_n = \frac{1}{1 + \mu} y_n \text{ to the sum } y_0 + y_1 + \dots + y_{n-1}.$$

Thus we can stop the drawing of curves, as soon as two consecutive ones y_n and y_{n+1} are found, the ordinates of which are proportional.

Though — generally — the above mentioned proportionality only appears exactly after an infinite number of iterations, it nevertheless will be approximately observed tolerably soon. Neglecting in such a case that part of the last found loading diagram which troubles the proportionality between its ordinates and those of the foregoing diagram, we can use the preceding remark, provided that 1° the neglected load diagram be insignificant, and 2° it gives no rise to following load diagrams which grow larger and larger.

The second condition is satisfied when $\frac{kl^4}{EI} < 14600$.

The justification of this latter statement can be given most naturally by the aid of the deductions, given by Mr. DROSTE. We therefore refer to his paper.

Mathematics. — “*An application of the theory of integral equations on the determination of the elastic curve of a beam, elastically supported on its whole length*”. By Dr. J. DROSTE. (Communicated by Prof. J. C. KLUIJVER).

(Communicated at the meeting of March 24, 1923).

1. Under the same title and at the same time a paper¹⁾ of Mr. BIEZENO appears in these Proceedings. The question, suggested, in N°. 4 of that paper as to the validity of the process of iteration used in it, will be answered here.

For that purpose we observe that the function of x , satisfying the differential equation

$$\frac{d^4 y}{dx^4} + \lambda y = q'(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the conditions at the ends of the interval, is a meromorphic function of λ . We might find it by means of the method of the variation of constants and then expand it in ascending powers of λ ; the radius of convergence R of the power series that stands after the first term (containing λ^{-1} as a factor) might easily be calculated then. After this it will be necessary to investigate whether it agrees or not for $\lambda = k'$ with the series of paper 1; it is only in the first case that the latter series will be valid for $k' < R$. For the sake of this investigation, however, and also in order to get an idea of the proportionality of the functions $\chi_n(x)$ (vid. I, 7), we prefer to use the method based upon the theory of the integral equation of FREDHOLM.

2. We construct a function of x , satisfying in the interval $(0, l)$ both the equation

$$\frac{d^4 y}{dx^4} + \lambda y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the conditions $y'' = y''' = 0$ at the ends, and being continuous as well as its first three derivatives everywhere in $(0, l)$ with the only exception of a saltus of the third derivative at the point ξ :

$$\left. \frac{d^3 y}{dx^3} \right|_{\xi+0} - \left. \frac{d^3 y}{dx^3} \right|_{\xi-0} = 0.$$

¹⁾ Referred to in the sequel as “paper I”.

This function we call $K(x, \xi, \lambda)$; it represents the deflexion of the beam, loaded by a load 1, which is concentrated at the point ξ .

Putting $\lambda = -\varrho^4$ the function

$$\mp \frac{1}{4\varrho^3} \{ \sinh \varrho (x - \xi) - \sin \varrho (x - \xi) \}$$

(the upper sign for $x \leq \xi$, the lower for $x \geq \xi$) will satisfy all conditions excepted those at the ends.

Assuming

$$K(x, \xi, \lambda) = \mp \frac{1}{4\varrho^3} \{ \sinh \varrho (x - \xi) - \sin \varrho (x - \xi) \} + \\ + A \cosh \varrho (x - \tfrac{1}{2}l) + B \sinh \varrho (x - \tfrac{1}{2}l) + C \cos \varrho (x - \tfrac{1}{2}l) + D \sin \varrho (x - \tfrac{1}{2}l),$$

we may determine A , B , C and D in such a way that $K(x, \xi, \lambda)$ satisfies the conditions at the ends. This gives

$$\begin{aligned} - A \cosh \tfrac{1}{2} \varrho l + B \sinh \tfrac{1}{2} \varrho l + C \cos \tfrac{1}{2} \varrho l - D \sin \tfrac{1}{2} \varrho l &= \frac{1}{4\varrho^3} \{ \sinh \varrho \xi + \sin \varrho \xi \}, \\ - A \sinh \tfrac{1}{2} \varrho l + B \cosh \tfrac{1}{2} \varrho l - C \sin \tfrac{1}{2} \varrho l - D \cos \tfrac{1}{2} \varrho l &= \frac{1}{4\varrho^3} \{ \cosh \varrho \xi + \cos \varrho \xi \}, \\ - A \cosh \tfrac{1}{2} \varrho l - B \sinh \tfrac{1}{2} \varrho l + C \cos \tfrac{1}{2} \varrho l + D \sin \tfrac{1}{2} \varrho l &= \\ &= \frac{1}{4\varrho^3} \{ \sinh \varrho (l - \xi) + \sin \varrho (l - \xi) \}, \\ - A \sinh \tfrac{1}{2} \varrho l - B \cosh \tfrac{1}{2} \varrho l - C \sin \tfrac{1}{2} \varrho l + D \cos \tfrac{1}{2} \varrho l &= \\ &= \frac{1}{4\varrho^3} \{ \cosh \varrho (l - \xi) + \cos \varrho (l - \xi) \}. \end{aligned}$$

Adding the first and the third of these equations and also the second and the fourth we get two equations containing only A and C . Subtracting the third from the first and the fourth from the second we get two equations containing only B and D . In this way we obtain

$$\begin{aligned} - A \cosh \tfrac{1}{2} \varrho l + C \cos \tfrac{1}{2} \varrho l &= \frac{1}{4\varrho^3} \{ \sinh \tfrac{1}{2} \varrho l \cosh \varrho (\xi - \tfrac{1}{2} \varrho l) + \sin \tfrac{1}{2} \varrho l \cos \varrho (\xi - \tfrac{1}{2} l) \}, \\ - A \sinh \tfrac{1}{2} \varrho l - C \sin \tfrac{1}{2} \varrho l &= \frac{1}{4\varrho^3} \{ \cosh \tfrac{1}{2} \varrho l \cosh \varrho (\xi - \tfrac{1}{2} l) + \cos \tfrac{1}{2} \varrho l \cos \varrho (\xi - \tfrac{1}{2} l) \}, \\ B \sinh \tfrac{1}{2} \varrho l - D \sin \tfrac{1}{2} \varrho l &= \frac{1}{4\varrho^3} \{ \cosh \tfrac{1}{2} \varrho l \sinh \varrho (\xi - \tfrac{1}{2} l) + \cos \tfrac{1}{2} \varrho l \sin \varrho (\xi - \tfrac{1}{2} l) \}, \\ B \cosh \tfrac{1}{2} \varrho l - D \sin \tfrac{1}{2} \varrho l &= \frac{1}{4\varrho^3} \{ \sinh \tfrac{1}{2} \varrho l \sinh \varrho (\xi - \tfrac{1}{2} l) - \sin \tfrac{1}{2} \varrho l \sin \varrho (\xi - \tfrac{1}{2} l) \}, \end{aligned}$$

From these equations A , B , C and D are easily solved; putting

$$\Delta_1(\varrho) = \cosh \frac{1}{2} \varrho l \sin \frac{1}{2} \varrho l + \sinh \frac{1}{2} \varrho l \cos \frac{1}{2} \varrho l,$$

$$\Delta_2(\varrho) = \cosh \frac{1}{2} \varrho l \sin \frac{1}{2} \varrho l - \sinh \frac{1}{2} \varrho l \cos \frac{1}{2} \varrho l,$$

we get

$$\begin{aligned} -4\varrho^3 \Delta_1(\varrho) \{ A \cosh \varrho (x - \frac{1}{2} l) + C \cos \varrho (x - \frac{1}{2} l) \} = \\ = (\cosh \frac{1}{2} \varrho l \cos \frac{1}{2} \varrho l + \sinh \frac{1}{2} \varrho l \sin \frac{1}{2} \varrho l) \cosh \varrho (x - \frac{1}{2} l) \cosh \varrho (\xi - \frac{1}{2} l) \\ + \cosh \varrho (x - \frac{1}{2} l) \cos \varrho (\xi - \frac{1}{2} l) + \cos \varrho (x - \frac{1}{2} l) \cosh (\xi - \frac{1}{2} l) \\ + (\cosh \frac{1}{2} \varrho l \cos \frac{1}{2} \varrho l - \sinh \frac{1}{2} \varrho l \sin \frac{1}{2} \varrho l) \cos \varrho (x - \frac{1}{2} l) \cos \varrho (\xi - \frac{1}{2} l), \end{aligned}$$

$$\begin{aligned} -4\varrho^3 \Delta_2(\varrho) \{ B \sinh \varrho (x - \frac{1}{2} l) + D \sin \varrho (x - \frac{1}{2} l) \} = \\ = (\cosh \frac{1}{2} \varrho l \cos \frac{1}{2} \varrho l - \sinh \frac{1}{2} \varrho l \sin \frac{1}{2} \varrho l) \sinh \varrho (x - \frac{1}{2} l) \sinh \varrho (\xi - \frac{1}{2} l) \\ + \sinh \varrho (x - \frac{1}{2} l) \sin \varrho (\xi - \frac{1}{2} l) + \sin \varrho (x - \frac{1}{2} l) \sinh \varrho (\xi - \frac{1}{2} l) \\ + (\cosh \frac{1}{2} \varrho l \cos \frac{1}{2} \varrho l + \sinh \frac{1}{2} \varrho l \sin \frac{1}{2} \varrho l) \sin \varrho (x - \frac{1}{2} l) \sin \varrho (\xi - \frac{1}{2} l). \end{aligned}$$

We now have calculated the function $K(x, \xi, \lambda)$; it appears to be a function with the denominator $4\varrho^3 \Delta_1(\varrho) \Delta_2(\varrho)$. The values of λ equating to zero this denominator are the characteristic numbers of the problem; as $K(x, \xi, \lambda)$ is symmetrical with respect to x and ξ that numbers will be all real. From this it follows that the corresponding values of ϱ have an argument that is a multiple of $\frac{1}{4}\pi$; it is easily proved to be an even multiple so that the values of ϱ will be real or purely imaginary and the corresponding values of λ negative or zero. For that purpose we first write $1 - \cosh \varrho l \cos \varrho l$ for $2\Delta_1(\varrho)\Delta_2(\varrho)$ and then substitute in it $\varrho l = \alpha + i\beta$; equating the real part to zero we get

$$\cosh \alpha \cosh \beta \cos \alpha \cos \beta + \sinh \alpha \sinh \beta \sin \alpha \sin \beta = 1,$$

which is not satisfied by $\beta = \pm \alpha \neq 0$, for substituting $\beta = \pm \alpha$ in it we get $\sinh^2 \alpha = \sin^2 \alpha$, which is impossible for $\alpha \neq 0$. Therefore the values of ϱ are real or purely imaginary and the characteristic numbers are *negative*, except one which is zero.

If ϱ be a root of $\Delta_1(\varrho) = 0$, also $i\varrho$ will be a root (and consequently $-\varrho$ and $-i\varrho$); the same is true with respect to the roots of $\Delta_2(\varrho) = 0$. We now call the positive roots of the equation

$$\tanh p = -\tanh p,$$

in the order of their magnitude p_1, p_2, \dots and the positive roots of the equation

$$\tanh p = \tanh p,$$

ordered in the same way q_1, q_2, \dots . Then the characteristic numbers will be

$$0, -\left(\frac{2p_n}{l}\right)^4, -\left(\frac{2q_n}{l}\right)^4, \dots (n = 1, 2, \dots)$$

3. We will also calculate the characteristic functions. If p represents one of the numbers p_n and q one of the numbers q_n we have to calculate the following limits:

$$\lim_{p \rightarrow 0} q^4 K(x, \xi, \lambda), \lim_{p \rightarrow 2p/l} \left\{ q^4 - \left(\frac{2p}{l} \right)^4 \right\} K(x, \xi, \lambda), \lim_{p \rightarrow 2q/l} \left\{ q^4 - \left(\frac{2q}{l} \right)^4 \right\} K(x, \xi, \lambda).$$

To none of the limits the term $\mp \frac{1}{4q^3} \{ \sinh q(x - \xi) - \sin q(x - \xi) \}$ contributes.

For the first of the limits we find immediately

$$\lim_{p \rightarrow 0} q^4 K(x, \xi, \lambda) = -\frac{1}{l} - \frac{12}{l^3} (x - \frac{1}{2}l)(\xi - \frac{1}{2}l).$$

To the second only the term $A \cosh q(x - \frac{1}{2}l) + C \cos q(x - \frac{1}{2}l)$ contributes. First we have

$$\lim_{p \rightarrow 2p/l} \frac{q^4 - (2p/l)^4}{4q^3 \Delta_1(q)} = \frac{1}{-l \cosh p \cos p}$$

and the numerator of the fraction we have found for

$$A \cosh q(x - \frac{1}{2}l) + C \cos q(x - \frac{1}{2}l)$$

changes for $q = 2p/l$ into

$$\begin{aligned} & \cosh 2p \left(\frac{x}{l} - \frac{1}{2} \right) \left\{ (\cosh p \cos p + \sinh p \sin p) \cosh 2p \left(\frac{\xi}{l} - \frac{1}{2} \right) + \cos 2p \left(\frac{\xi}{l} - \frac{1}{2} \right) \right\} \\ & + \cos 2p \left(\frac{x}{l} - \frac{1}{2} \right) \left\{ \cosh 2p \left(\frac{\xi}{l} - \frac{1}{2} \right) + (\cosh p \cos p - \sinh p \sin p) \cos 2p \left(\frac{\xi}{l} - \frac{1}{2} \right) \right\} \end{aligned}$$

From $\cosh p \sin p + \sinh p \cos p = 0$ we have

$$\cosh p \cos p - \sinh p \sin p = \frac{\cosh p}{\cos p},$$

$$\cosh p \cos p + \sinh p \sin p = \frac{\cos p}{\cosh p},$$

and consequently the numerator becomes

$$\left\{ \cosh 2p \left(\frac{x}{l} - \frac{1}{2} \right) + \frac{\cosh p}{\cos p} \cos 2p \left(\frac{x}{l} - \frac{1}{2} \right) \right\} \frac{\cos p}{\cosh p} \cosh 2p \left(\frac{\xi}{l} - \frac{1}{2} \right) + \cos 2p \left(\frac{\xi}{l} - \frac{1}{2} \right) \left\{ \cosh 2p \left(\frac{\xi}{l} - \frac{1}{2} \right) + \frac{\cosh p}{\cos p} \cos 2p \left(\frac{\xi}{l} - \frac{1}{2} \right) \right\}$$

In this way we find

$$\lim_{p \rightarrow 2p/l} \left\{ q^4 - \left(\frac{2p}{l} \right)^4 \right\} K(x, \xi, \lambda) =$$

$$-\frac{1}{l} \left\{ \frac{\cosh 2p \left(\frac{x}{l} - \frac{1}{2} \right)}{\cosh p} + \frac{\cos 2p \left(\frac{x}{l} - \frac{1}{2} \right)}{\cos p} \right\} \left\{ \frac{\cosh 2p \left(\frac{\xi}{l} - \frac{1}{2} \right)}{\cosh p} + \frac{\cos 2p \left(\frac{\xi}{l} - \frac{1}{2} \right)}{\cos p} \right\}.$$

In the same way

$$\lim_{p \rightarrow 2q/l} \left\{ e^4 - \left(\frac{2q}{l} \right)^4 \right\} K(x, \xi, \lambda) =$$

$$-\frac{1}{l} \left\{ \frac{\sinh 2q \left(\frac{x}{l} - \frac{1}{2} \right)}{\sinh q} + \frac{\sin 2q \left(\frac{x}{l} - \frac{1}{2} \right)}{\sin q} \right\} \left\{ \frac{\sinh 2q \left(\frac{\xi}{l} - \frac{1}{2} \right)}{\sinh q} + \frac{\sin 2q \left(\frac{\xi}{l} - \frac{1}{2} \right)}{\sin q} \right\}.$$

Putting

$$\varphi_0(x) = \frac{1}{\sqrt{l}}, \quad \varphi_n(x) = \frac{1}{\sqrt{l}} \left\{ \frac{\cosh 2p_n \left(\frac{x}{l} - \frac{1}{2} \right)}{\cosh p_n} + \frac{\cos 2p_n \left(\frac{x}{l} - \frac{1}{2} \right)}{\cos p_n} \right\},$$

$$\psi_0(x) = \frac{2\sqrt{3}}{l\sqrt{l}} \left(x - \frac{1}{2} \right), \quad \psi_n(x) = \frac{1}{\sqrt{l}} \left\{ \frac{\sinh 2q_n \left(\frac{x}{l} - \frac{1}{2} \right)}{\sinh q_n} + \frac{\sin 2q_n \left(\frac{x}{l} - \frac{1}{2} \right)}{\sin q_n} \right\},$$

$$(n = 1, 2, \dots)$$

the functions $\varphi_n(x)$, $\psi_n(x)$ ($n = 0, 1, 2, \dots$) will be the orthogonal and normal characteristic numbers; they satisfy equation (2), λ being replaced by the corresponding characteristic number.

Now drawing graphs of the functions $y = tg x$, $y = tgh x$ and $y = -tgh x$ in one figure, it is easily seen that p_n is an angle in the $2n^{\text{th}}$ quadrant, and q_n an angle in the $(2n+1)^{\text{th}}$ quadrant. For $n \rightarrow \infty$ p_n and q_n converge to the middlepoints of the intervals. From this it follows that $\cos p_n$ and $\sin p_n$ converge to $\pm \frac{1}{2}\sqrt{2}$ and it is easily seen that the absolute value of $\varphi_n(x)$ and $\psi_n(x)$ remains less than a number which is independent from x and n . Now as

$$\lim_{n \rightarrow \infty} \frac{p_n}{n} = \lim_{n \rightarrow \infty} \frac{q_n}{n} = \pi$$

the two series occurring in

$$K(x, \xi, \lambda) = \frac{\varphi_0(x) \varphi_0(\xi) + \psi_0(x) \psi_0(\xi)}{\lambda} + \sum_{n=1}^{\infty} \frac{\varphi_n(x) \varphi_n(\xi)}{\lambda + (2p_n/l)^4} +$$

$$+ \sum_{n=1}^{\infty} \frac{\psi_n(x) \psi_n(\xi)}{\lambda + (q_n/l)^4} \dots \dots \dots (3)$$

will be uniformly convergent and the right hand side therefore will be equal to $K(x, \xi, \lambda)$.

4. We now suppose y to be the required solution of (1), viz. that solution for which $y'' = y''' = 0$ in the points $x = 0$ and $x = l$ and which is continuous in $(0, l)$ as well as its first three derivatives; as to y'''' it may have a saltus in a finite number of points a_i , which will be the case if $q'(x)$ has in the points a_i discontinuities for which $q'(a_i + 0)$ and $q'(a_i - 0)$ exist. The points a_i and the value ξ divide the interval $(0, l)$ into a number of subintervals; in the interior of each of them we have

$$\begin{aligned} \frac{d}{dx} [y'''' K(x, \xi, \lambda) - y'' K'(x, \xi, \lambda) + y' K''(x, \xi, \lambda) - y K''''(x, \xi, \lambda)] = \\ = y'''' K(x, \xi, \lambda) - y K''''(x, \xi, \lambda). \end{aligned}$$

Integrating the equation over the subintervals, adding the results and regarding that $y'' = y''' = K''(x, \xi, \lambda) = K''''(x, \xi, \lambda) = 0$ for $x = 0$ and $x = l$ and that $y, y', y'', y''', K, K', K'', K'''$ are continuous everywhere except K'''' in ξ , we find

$$-y(\xi) = \int_0^l \{y'''(x) K(x, \xi, \lambda) - y(x) K''''(x, \xi, \lambda)\} dx.$$

Replacing y'''' by $q' - \lambda y$ from (1) and K'''' by $-\lambda K$ from (2) we get

$$y(\xi) = \int_0^l K(x, \xi, \lambda) q'(x) dx$$

or interchanging x and ξ and observing the symmetry of $K(x, \xi, \lambda)$ with respect to x and ξ

$$y(x) = \int_0^l K(x, \xi, \lambda) q'(\xi) d\xi.$$

If the beam is not loaded by $q(x)$, but by N loads Q_i , concentrated in the points ξ_i , we have

$$y(x) = \sum_{i=1}^N Q_i K(x, \xi_i, \lambda)$$

where $Q_i = Q_i/EI$. If the beam bears both the load $q(x)$ and the loads Q_i we have

$$y(x) = \int_0^l K(x, \xi, \lambda) q'(\xi) d\xi + \sum_{i=1}^N Q_i K(x, \xi_i, \lambda) \quad (4)$$

From (4) it follows that y is a meromorphic function of λ with the poles 0 , $-(2p_{n/l})^4$ and $-(2q_{n/l})^4$. This is easily seen from sub-

stituting (3) in (4) and integrating term by term, which is permitted, the series (3) being uniformly convergent.

Expanding y in a series of ascending powers of λ (the first term will in general contain λ^{-1}) the expansion will generally be convergent for $|\lambda| < (2p_1/l)^4$; only if the term with the denominator $\lambda + (2p_1/l)^4$ cancels, the expansion will be valid for larger values of λ . In case the Q_i 's are zero this occurs if $q(x)$ be orthogonal to $\varphi_1(x)$. We thus see that if the expansion of paper 1 be exact and if not by chance

$$\int_0^l q(x) \varphi_1(x) dx = 0$$

it converges only if

$$\frac{kl^4}{EI} \leq (2p_1)^4 = 500,54665 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From (4) we deduce a formula which will be of use further on. Supposing the beam to bear only a load $p(x)$ pro unit of length and to be in equilibrium, we will have

$$\int_0^l p(x) dx = \int_0^l x p(x) dx = 0$$

or which is the same

$$\int_0^l p(x) \varphi_0(x) dx = \int_0^l p(x) \psi_0(x) dx = 0.$$

Now from (4), in which $q'(x)$ is to be replaced by $p(x)/EI$ and in which $Q_i = 0$, we have

$$y(x) = \int_0^l \bar{K}(x, \xi, \lambda) p(\xi) d\xi,$$

where $\bar{K}(x, \xi, \lambda)$ arises from $K(x, \xi, \lambda)$ by omitting in (3) the term with the denominator λ . Putting $\lambda = 0$ $\bar{K}(x, \xi, \lambda)$ changes into

$$K(x, \xi) = \sum_{n=1}^{\infty} \frac{\varphi_n(x) \varphi_n(\xi)}{(2p_{n/l})^4} + \sum_{n=1}^{\infty} \frac{\psi_n(x) \psi_n(\xi)}{(2q_{n/l})^4}, \quad . \quad . \quad . \quad (6)$$

and we get

$$y(x) = \frac{1}{EI} \int_0^l K(x, \xi) p(\xi) d\xi \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This represents the deflection of the beam under the conditions that the beam be in equilibrium and that the ground be absent; it is such that

$$\int_0^l y(x) dx = \int_0^l x y(x) dx = 0 \quad \dots \quad (8)$$

since $K(x, \xi)$ is orthogonal with respect to $\varphi_0(x)$ and $\psi_0(x)$. By the conditions (8) the deflection is perfectly determined and (7) represents it.

5. We shall now prove that the series deduced from (4) agrees for $\lambda = k'$ with the series of paper I. Representing the iterations of $K(x, \xi)$ by $K_1(x, \xi)$, $K_2(x, \xi)$. . . we get for $|\lambda| < (2\rho_1/l)^4$

$$\begin{aligned} \overline{K}(x, \xi, \lambda) &= K(x, \xi) - \lambda K_1(x, \xi) + \lambda^2 K_2(x, \xi) \dots, \\ \int_0^l \overline{K}(x, \xi, \lambda) q'(\xi) d\xi &= \int_0^l K(x, \xi) q'(\xi) d\xi - \lambda \int_0^l K_1(x, \xi) q'(\xi) d\xi + \dots, \end{aligned}$$

as is proved in the theory of integral equations. From this it follows that (4) for $|\lambda| < (2\rho_1/l)^4$ takes the form

$$y(x) = y_0(x) + y_1(x) + y_2(x) + \dots, \quad (9)$$

where

$$\begin{aligned} y_0(x) &= \frac{1}{k} \left\{ \varphi_0(x) \int_0^l \varphi_0(\xi) q(\xi) d\xi + \psi_0(x) \int_0^l \psi_0(\xi) q(\xi) d\xi + \right. \\ &\quad \left. + \varphi_0(x) \sum_{i=1}^N Q_i \varphi_0(\xi_i) + \psi_0(x) \sum_{i=1}^N Q_i \psi_0(\xi_i) \right\} \\ y_1(x) &= \frac{1}{EI} \int_0^l K(x, \xi) q(\xi) d\xi + \frac{1}{EI} \sum_{i=1}^N Q_i K(x, \xi_i), \\ y_2(x) &= -k' \left\{ \frac{1}{EI} \int_0^l K_1(x, \xi) q(\xi) d\xi + \frac{1}{EI} \sum_{i=1}^N Q_i K_1(x, \xi_i) \right\} = \\ &= -\frac{k}{EI} \int_0^l K(x, \xi) y_1(\xi) d\xi, \\ &\dots \end{aligned}$$

$$y_{n+1}(x) = -\frac{k}{EI} \int_0^l K(x, \xi) y_n(\xi) d\xi,$$

.....

Each of the functions $y_n(x)$, except $y_0(x)$, satisfies (8). We shall now prove the terms y_0, y_1, y_2, \dots to be the same as the corresponding quantities of I, 5, from which it will follow that the series $y_0 + y_1 + \dots$ agrees with the series of I, 4. Indeed in the first place $y_0(x)$ is a linear function of x ; the function $ky_0(x)$ represents the linear load $\alpha x + \beta$, which is defined in I, 5 and is statically equivalent to the given load. For we have

$$k \int_0^l y_0(x) \varphi_0(x) dx = \int_0^l \varphi_0(\xi) q(\xi) d\xi + \sum_{i=1}^N Q_i \varphi_0(\xi_i),$$

$$k \int_0^l y_0(x) \psi_0(x) dx = \int_0^l \psi_0(\xi) q(\xi) d\xi + \sum_{i=1}^N Q_i \psi_0(\xi_i),$$

or substituting in it the expressions found for the functions $\varphi_0(x)$ and $\psi_0(x)$

$$\int_0^l ky_0(x) dx = \int_0^l q(\xi) d\xi + \sum_{i=1}^N Q_i$$

$$\int_0^l (x - \frac{1}{2}) ky_0(x) dx = \int_0^l (\xi - \frac{1}{2}) q(\xi) d\xi + \sum_{i=1}^N Q_i (\xi_i - \frac{1}{2}),$$

which proves the proposition.

Omitting from (9) the deflexion y_0 , the remaining terms represent the remaining deflexion. This becomes y_1 for $k=0$ and so y_1 represents the deflexion which the beam, if not supported by the ground, gets under the influence of the load that remains after subtraction of $\alpha x + \beta$ from the given load. As besides $y_1(x)$ satisfies (8), it is identical with the quantity y_1 of I, 5.

The reaction of the ground, arising from the deflexion y_1 , represents a load $-ky_1$ of the beam; by this load the beam, if not supported by the ground, would get a deflexion, which we may calculate from (7) viz.

$$-\frac{k}{EI} \int_0^l K(x, \xi) y_1(\xi) d\xi.$$

This represents the deflexion $y_2(x)$; it is seen to be the same as the quantity y_2 of I, 5. In the same way we continue and so we may prove that (9) agrees term by term with the series of paper I.

7. In case the expansion do not converge, it may happen that the method of graphical integration, communicated in paper I, remains still valid (vid I, 7); this depends on the approximate proportionality of the functions $y_n(x)$ for large values of n . We shall prove this now; more exactly: we shall prove

$$\lim_{n \rightarrow \infty} \frac{y_{n+1}(x)}{y_n(x)} = -\mu$$

where μ is independent from x .

Now $K_n(x, \xi)$ is represented by the absolutely and uniformly convergent series

$$K_n(x, \xi) = \sum_{m=1}^{\infty} \frac{w_m(x) w_m(\xi)}{\lambda_m^n},$$

where the quantities λ_m represent the numbers $(2\rho_i/l)^4$ and $(2q_i/l)^4$ in the order of their magnitude and the functions $w_m(x)$ are the corresponding normal orthogonal functions. Putting

$$\int_0^l w_m(\xi) q'(\xi) d\xi + \sum_{m=1}^N Q'_i w_m(\xi_i) = P_m, \quad \dots \quad (10)$$

we get the absolutely and uniformly convergent series

$$y_n(x) = (-k')^{n-1} \sum_{m=1}^{\infty} \frac{P_m w_m(x)}{\lambda_m^n} \quad (n = 1, 2, \dots)$$

Supposing h to be the smallest value of m for which $P_m \neq 0$, we can write

$$y_n(x) = \frac{(-k')^{n-1}}{\lambda_h^{n-1}} \left\{ P_h w_h(x) + \left(\frac{\lambda_h}{\lambda_{h+1}} \right)^n \sum_{m=1}^{\infty} \left(\frac{\lambda_{h+1}}{\lambda_{h+m}} \right)^n P_m w_m(x) \right\}.$$

The series in the right hand member of this equation has an absolute value which is less than the sum of the series

$$\sum_{m=1}^{\infty} \frac{\lambda_{h+1}}{\lambda_{h+m}} |P_m w_m(x)|,$$

a quantity which is independent from n . From this and from

$$\lim_{n \rightarrow \infty} \left(\frac{\lambda_h}{\lambda_{h+1}} \right)^n = 0$$

we get

$$\lim_{n \rightarrow \infty} \frac{\lambda_h^n}{(-k')^{n-1}} y_n(x) = P_h w_h(x)$$

In this way we find

$$\lim_{n \rightarrow \infty} \frac{y_{n+1}(x)}{y_n(x)} = -\frac{k'}{\lambda_h} \lim_{n \rightarrow \infty} \frac{\frac{\lambda_h^{n+1}}{(-k')^n} y_{n+1}(x)}{\frac{\lambda_h^n}{(-k')^{n-1}} y_n(x)} = -\frac{k'}{\lambda_h},$$

which proves the proposition; we see that

$$\mu = -\frac{k'}{\lambda_h}.$$

Now, if in drawing the successive deflexions y_0, y_1, y_2, \dots it is found that $y_{n+1}:y_n$ is sufficiently independent from x , it will be permitted occasionally to consider

$$\bar{y}_n = y_0 + y_1 + \dots + y_{n-1} + \frac{y_n}{1 + \frac{k'}{\lambda_h}}$$

to be the deflexion y . For we have

$$\begin{aligned} y_1 + y_2 + \dots + y_{n-1} + \frac{y_n}{1 + \frac{k'}{\lambda_h}} &= \sum_{v=1}^{n-1} (-k')^{v-1} \sum_{m=h}^{\infty} \frac{P_m w_m(x)}{\lambda_m^v} + \frac{y_n}{1 + \frac{k'}{\lambda_h}} \\ &= \sum_{m=h}^{\infty} \frac{P_m w_m(x)}{\lambda_m + k'} \left\{ 1 - \left(-\frac{k'}{\lambda_m} \right)^{n-1} \right\} + \sum_{m=h}^{\infty} \frac{P_m w_m(x)}{\lambda_m^n} \cdot \frac{(-k')^{n-1}}{1 + \frac{k'}{\lambda_h}} \\ &= \sum_{m=h}^{\infty} \frac{P_m w_m(x)}{\lambda_m + k'} - \sum_{m=h}^{\infty} P_m w_m(x) \left(\frac{-k'}{\lambda_m} \right)^{n-1} \left\{ \frac{1}{\lambda_m + k'} - \frac{\lambda_h}{\lambda_m(\lambda_h + k')} \right\}. \end{aligned}$$

and as

$$y = y_0 + \sum_{m=1}^{\infty} \frac{P_m w_m(x)}{\lambda_m + k'}$$

we get

$$\bar{y}_n - y = - \sum_{m=h+1}^{\infty} P_m w_m(x) \left(\frac{-k'}{\lambda_m} \right)^{n-1} \left\{ \frac{1}{\lambda_m + k'} - \frac{\lambda_h}{\lambda_m(\lambda_h + k')} \right\},$$

since $m = h$ gives zero. If $k' < \lambda_{h+1}$, the series has zero as a limit for $n \rightarrow \infty$, which is easily seen by writing it in the form

$$\bar{y}_n - y = - \left(\frac{-k'}{\lambda_{h+1}} \right)^{n-1} \sum_{m=h+1}^{\infty} P_m w_m(x) \left(\frac{\lambda_{h+1}}{\lambda_m} \right)^{n-1} \left\{ \frac{1}{\lambda_m + k'} - \frac{\lambda_h}{\lambda_m(\lambda_h + k')} \right\}$$

since the absolute values of the series occurring in the right hand member is less than the sum of the convergent series

$$\sum_{m=h+1}^{\infty} \frac{|P_m w_m(x)|}{\lambda_m + k'}$$

It thus appears that we may consider $\overline{y_n}$ to be the required deflexion y , supposed n be large enough and $k' < \lambda_{h+1}$. If g is, after h , the first value of m such that $P_m \neq 0$, the condition $k' < \lambda$ must be satisfied if we wish to replace y by $\overline{y_n}$ for large values of n .

Chemistry. — “*The Phenomenon of Electrical Supertension.*” III. ¹⁾
By Prof. A. SMITS. (Communicated by Prof. P. ZEEMAN.)

(Communicated at the meeting of February 24, 1923.)

In my book “*Die Theorie der Allotropie*”²⁾, and also in the preceding communications I have treated the electrical supertension only very briefly. Therefore I will discuss this important phenomenon somewhat more at length here.

We imagine the case that a palladium or platinum electrode is made cathode. For the explanation of the phenomenon that will now

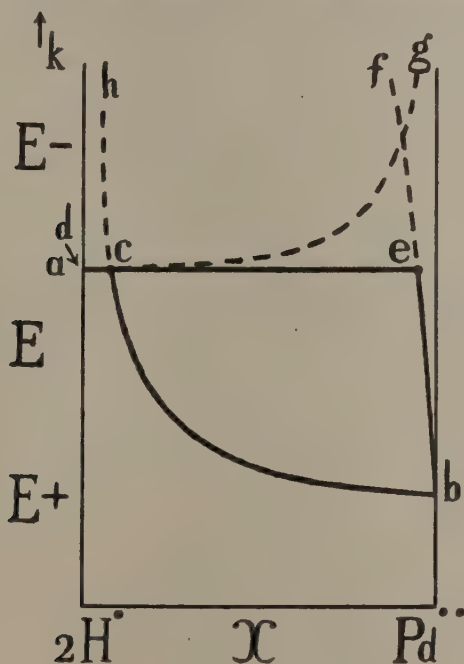


Fig. 1.

appear, we shall make use of the E, X -diagram, in which the experimental electric potential of the electrodes is plotted as function

¹⁾ These Proc. Vol. XXI No. 3, p. 375 (1918); Vol. XXI, No. 8, p. 1106 (1919).

²⁾ JOHANN AMBROSIUS BARTH, Leipzig. 1921.

English edition LONGMANS, GREEN and Co. London 1922.

French edition GAUTHIER VILLARS. Paris. 1923.

of the concentration; on the assumption that the pressure (1 atm.), temperature, and total ion-concentration (metal ions + hydrogen ions) are constant. In the foregoing figure 1 hydrogen is taken for one electrode, and palladium for the other, but instead of the latter platinum might, of course, have been chosen just as well.

Line bh indicates the potentials of the series of electrolytes that can coexist with different palladium phases. These phases of the palladium are different, because palladium dissolves the hydrogen in quantities which increase with the hydrogen-ion concentration of the electrolyte.

Line bf indicates the potentials of the different palladium phases containing hydrogen¹⁾, which coexist with the different electrolytes. In our E, X -figure the potential of the metal-phase can be read on the E -axis, but it is clear that on this axis also the potential of the electrolyte can be read, when we reverse the sign.

The line ag represents the potentials of the different electrolytes coexisting with the gaseous hydrogen phases. These hydrogen phases consist of pure hydrogen, and lie, therefore, on the hydrogen axis. Accordingly the portion ak of the hydrogen axis gives the potentials of the hydrogen phases coexisting with the different electrolytes.

The point of intersection c of the lines bh and ag represents the electrolyte which can coexist at the same time with the palladium phase (e) and with the hydrogen phase (d), so that it also shows the potential of this three-phase equilibrium. The situation of this point of intersection follows from the solubility products of hydrogen and palladium:²⁾

$$L_{H_2} = (H^*)^2 (\theta)^2 = 10^{2 \times -48}$$

$$L_{Pd} = (Pd'') (\theta)^2 = 10^{2 \times -62.2}.$$

At the three-phase equilibrium

$$(\theta)_{H_2} = (\theta)_{Pd}$$

from which follows:

$$\frac{(Pd'')}{(H^*)^2} = \frac{L_{Pd}}{L_{H_2}} = 10^{2 \times -14.2}.$$

If (H^*) is put = 1, then $(Pd'') = 10^{2 \times 14.2}$.

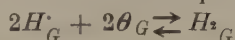
From this it is seen that the point e lies very much on one side, and that when a palladium electrode was immersed in a 1-N sulphuric

¹⁾ This line indicates the gross hydrogen concentrations, and gives, therefore, no information about the state in which the hydrogen is.

²⁾ Compare with regard to the smallness of these products the remarks in "The Theory of Allotropy" in the chapter: "Small concentrations" p. 172.

acid solution, and the palladium was and remained in inner equilibrium, this metal would dissolve a little, till the palladium concentration of $10^2 \times 14.2$ was reached, while a corresponding inappreciable quantity of hydrogen would have been generated. In this it is assumed that both platinum and hydrogen continue to be in inner equilibrium, for the value used for L_H , agrees with the value for hydrogen in inner equilibrium, and we shall for the moment assume the value used for L_{Pd} also to agree with the condition of inner equilibrium of Pd . Pd is, however, an inert metal, so that the solubility product of this metal will in reality have decreased through the slight attack, and the dissolving will have already stopped, before the palladium ion concentration $10^2 \times 15.2$ has been reached ¹⁾.

For the sake of simplicity we shall, however, assume here that no disturbance of the Pd takes place, and that the three-phase equilibrium is established, in which the Pd -phase e coexists with the electrolyte c and with the hydrogen phase d at a pressure of one atmosphere. When now the Pd -electrode is made cathode, or in other words, when electrons are added to the Pd , hydrogen and palladium ions in the ratio of $1 : 10^2 \times 14.2$ or practically only hydrogen ions will be separated at this electrode. It will now depend on the velocity with which the inner equilibrium



sets in, if the hydrogen formed will coexist in a state of internal equilibrium or in a state of formation. In this condition the solubility product of the hydrogen is greater, and the point that now denotes the coexisting hydrogen phase, will lie on a potential curve that lies at more negative values, and is represented by $a'g'$ in fig. 2. We must, however, not forget that this line could only be realised when the state of formation of the hydrogen discussed just now could coexist unchanged in electro-motive equilibrium with a series of solutions. This is, however, not the case; only one point can be realised on this curve, and this is the point indicating the liquid layer that coexists with the hydrogen phase d' , which is in a state of formation, and with the palladium phase e' . The heterogeneous equilibrium between the metal boundary layer and the hydrogen boundary layer, just as that with the liquid boundary layer, having been immediately established, the palladium boundary layer will also contain too many hydrogen ions and electrons, which means that also the hydrogen dissolved in this metal boundary layer, will be in a state of formation.

¹⁾ The potential $+ 0.82$ V., from which the solubility product $L_{Pd} = 10^2 \times 62.2$ has been calculated, is most probably already a potential of a disturbed state of the metal palladium.

We may, of course, also start from the *Pd*, and say that only in the *Pd*-electrode to which electrons are added, and in which hydrogen ions dissolve, hydrogen is formed in a state of formation, and that *afterwards* gaseous hydrogen occurs in a state of formation, but this only implies a difference so far as the first moments are concerned, for when once electrolytic generation of hydrogen has set in, this

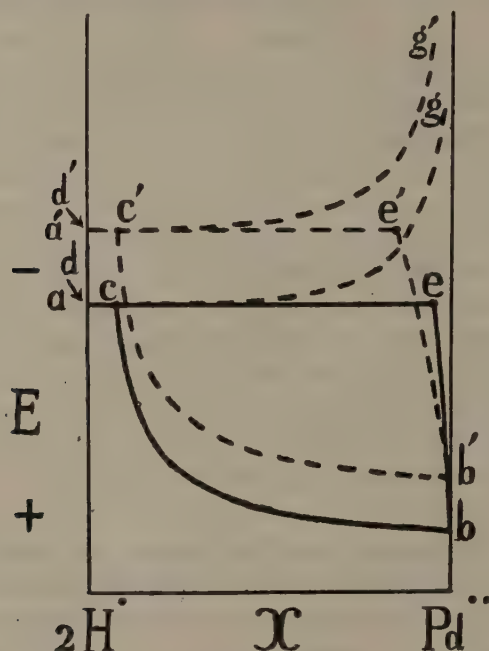


Fig. 2.

will occur in a state of formation at the same time in the gas phase and in the metal phase.

It should be pointed out here that when we have a homogeneous phase, as the solid solution of hydrogen in palladium, the electrical potential of these two components with respect to the coexisting electrolyte must be the same. This applies also to the solid solutions lying on the line *be*, but in the solid solution lying on this line there is equilibrium between hydrogen molecules, hydrogen ions, and electrons, whereas this is not the case in the *Pd*-boundary layer which coexists with hydrogen in a state of formation.

This is, therefore, the reason that the *Pd*-phase *e'* coexisting with the hydrogen phase *d'*, does not lie on the prolongation of the line *be*.

The hydrogen dissolved in the *Pd*-phase *e'* is in the state of formation, and consequently this phase is richer in hydrogen ions and electrons than when the hydrogen is in inner equilibrium. The

potential of the dissolved hydrogen in e' is more strongly negative, and the same must, therefore, hold for the Pd . It is now, however, the question in what way the potential of the palladium has undergone this change.

It is clear that the Pd must have become richer in Pd -ions and electrons. We have already seen that this phase has become richer in electrons through addition of hydrogen in a state of formation, so that only the question is still to be answered how the concentration of the Pd -ions can have been increased. This must have taken place through the reaction



in which, therefore, hydrogen ions have ceded their charge to Pd -atoms. We, therefore, come to the conclusion that the palladium boundary layer, which coexists with hydrogen in a state of formation, will possess too many hydrogen ions, palladium ions and electrons, or in other words, that it will contain both hydrogen and palladium in a state of formation.

If palladium could coexist in the same state of formation with a 1.N. solution of a palladium salt, the electric potential would, of course, possess a more strongly negative value than corresponds to point b in fig. 2. This more strongly negative potential is indicated by b' . And when, therefore, the same state of formation of Pd could continue to exist also in contact with the whole series of solutions, the line $b'e'$ would indicate the solid solutions which can coexist with the electrolytes lying on the line $b'c'$. The new three-phase equilibrium that is found when Pd is made cathode at a definite density of current, and in which hydrogen escapes in a state of formation, is denoted by the points $d'c'e'$. The line $a'c'g'$ rising very little throughout the greater part of the concentration region, it is clear that the value of the negative potential in this new three-phase equilibrium would be equally great when the point c' lay on the prolongation of the line bc , and the point e on the prolongation of the line be , but as we demonstrated above, the points c' and e' belong to other lines than those that are mentioned here. It follows from these considerations that in the case of electrolytic generation of hydrogen the state of formation of the hydrogen in the coexisting hydrogen and palladium phases are *very closely* related. This makes it clear that the cathode metal can exert influence on the degree of super-tension. The state of formation is a state of non-equilibrium, and the different cathode metals will, to a different degree, accelerate the conversion of this state of non-equilibrium in the direction of the inner

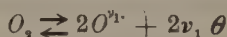
equilibrium. This is the reason why the so-called super-tension of hydrogen is different, when different metal cathodes are used.

It is self-evident that when the state of formation of the hydrogen does not vanish too quickly, the hydrogen must possess an abnormally high conductivity for electricity immediately after the escape. This phenomenon was, indeed, found long ago¹⁾, but it was tried to explain it in another way; it is, however, probable that this phenomenon is for the greater part to be attributed to the state of formation.

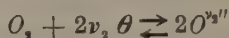
The activity of the hydrogen dissolved in the metal phase, is in perfect harmony with the considerations given here. As regards the temporary variations of the super-tension, they will have to be explained by the slow change in constitution of the coexisting phases. The heterogeneous equilibrium between the boundary layers is established with great velocity, but the composition of the phases changes slowly, and this must be the reason that the three-phase equilibrium metal-electrolyte-hydrogen changes slowly.

In conclusion I will still point out that analogous considerations, of course, apply to oxygen and other non-metals. As is discussed in "The Theory of Allotropy" p. 160 et seq, the extension of this theory to non-metals, necessitated the assumption that the atoms of all elements can split off and receive electrons.²⁾ The difference between the solubilities of the positive and the negative ions in elements with pronounced metal- resp. metalloid character, is so great that for the explanation of the electro-motive behaviour as a rule only the positive or the negative ions need be taken into account. But as was also already stated the supposition mentioned must very certainly be used when the positive charges of non-metals with regard to electrolytes, and likewise the small electric conductivity of non-metals in electrically neutral condition, is to be explained. Further the said supposition is also required to make clear the formation of compounds between metals.³⁾

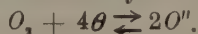
When we now return to the non-metals and choose oxygen as example, we have to consider the two following reactions:



and



As $v_2 = 2$, the latter reaction may be written:



¹⁾ BECKER. Jahrb. der Radioaktivität. 9, 52 (1912).

²⁾ Theory of Allotropy p. 160.

The latter equation is sufficient to explain the electric super-tension of the oxygen. It was stated ¹⁾ that in the case of anodic polarisation of an unattackable electrode or an inert metal the separated oxygen must relatively contain too few electrons and too few negative oxygen ions, so that oxygen in a state of formation or in other words oxygen in super-tension would have to possess an abnormally small electric conductivity immediately after its formation, at least when no other phenomena neutralise this effect.

When we have an inert metal, i. e. a metal that can be easily disturbed, and we make this anode, polarisation will take place. If the disturbance of the metal goes so far that oxygen is separated, then, the metal boundary layer being poor in ions and electrons, also the coexisting oxygen phase will be abnormally poor in electrons. Besides the other substances coexisting in the liquid, the metal boundary layer will also contain oxygen dissolved, and it is evident that the state of this oxygen, dissolved in the metal, will depend on the state of the oxygen in the coexisting oxygen layer.

*Laboratory for General and Inorganic
Chemistry of the University.*

Amsterdam, Februari 1923.

¹⁾ Theory of Allotropy p. 164.

Chemistry. — "*The Influence of Intensive Drying on Internal Conversion*". I. By Prof. A. SMITS. (Communicated by Prof. P. ZEEMAN).

(Communicated at the meeting of March 24, 1923).

In December 1921 a communication was published in the 100th volume of the *Z. f. physik. Chemie* under the same title as is given above. In manuscript this communication was at first more extensive, for it also contained a possible explanation of the great influence found by BAKER of intensive drying on the chemical reactivity of gases, and besides a discussion of the sa-ammoniac problem¹⁾. The reason why for the present I withheld this part was as follows.

I was at the time still in doubt whether in intensive drying it should be assumed that a fixation or a shifting of the inner equilibrium takes place. The results of BAKER's researches²⁾ published then spoke greatly in favour of a shifting, but at first this assumption seemed open to objections, because it is then necessary to assume that the slightest trace of moisture can give rise to a great displacement of the inner equilibrium.

Afterwards, when BAKER had published³⁾ a new series of experiments, it seemed nevertheless the most probable conclusion that here a shifting of the inner equilibrium takes place, which from a thermodynamic standpoint means that very much work is required to withdraw the last traces of water from a system.

Accordingly I showed in the English and in the French edition of the *Theory of Allotropy*, in which I devoted a chapter to BAKER's experiments, that in my opinion intensive drying gives rise to a displacement of the internal equilibrium. Since then my own investigation, which I carried out with some of my pupils, has confirmed this supposition.

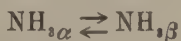
The explanation of the influence of intensive drying on reactivity, which I left unpublished so far, is exceedingly simple, for we

¹⁾ Also the influence of intensive drying on the properties of Sal ammoniac, becomes explicable, when this substance is assumed to contain two kinds of molecules, one of which is dissociable, and the other is not.

²⁾ *Trans. Chem. Soc.* **51**, 2339 (1903).

³⁾ *Trans. Chem. Soc.* **121**, 568 (1922).

have only to apply the theory of allotropy, i.e. we have to assume that every phase of these substances contains at least two different kinds of molecules, which are of course in inner equilibrium in the case of unary behaviour, to which we add the supposition that at least one of these kinds of molecules is chemically inactive. This is very well possible, since the mechanism of the transformation into another type of molecule will be an entirely different one from that of chemical action with other substances. To represent the case as simply as possible we can then assume that there are only two different kinds of molecules, one of which is active, the other inactive. When for ammonia we denote them by $\text{NH}_3\alpha$ and $\text{NH}_3\beta$, we have in each phase in the case of unary behaviour, the following inner equilibrium:



My supposition was this that on intensive drying this inner equilibrium is shifted towards the inactive side, and in this case, *completely*, so that in the ammonia remains that only contains the inactive kind of molecules.

I will just mention here that I emphatically pointed out before that the expression "different kinds of molecules" should be taken in its widest sense. It should comprise not only the isomer and polymer molecules, but also the electrically charged dissociation products, ions + electrons, and it stands to reason that in many cases the difference between the different kinds of molecules lies in a difference in the atomic structure.

It is particularly the more recent views of atomic structure that have brought to light that between the different atoms very subtle differences are possible, which are e.g. in connection with a change of the quanta values of the valency-electron-paths, and this leads to kinds of molecules with more subtle differences than those which are assumed to exist between the ordinary isomers. The fact, however, remains that also these different kinds of molecules may be ranged under this category when the sense in which the idea "isomery" is taken, is very wide.

During my investigation there appeared a publication by BALY and DUNCAN¹⁾, in which they communicate among other things that the rapidity at which gaseous ammonia, withdrawn from an iron cylinder with liquid ammonia, is decomposed by a platinum spiral heated at a definite temperature, is dependent on the velocity of evaporation of the liquid ammonia. On rapid evaporation ammoniac gas

¹⁾ J. Chem. Soc. **121** en **122**, 1003 (1922).

was obtained of much smaller velocity of decomposition than on slow evaporation. BALY and DUNCAN expressed the opinion that this difference is probably caused by this, that on rapid evaporation there is formed a gas phase rich in the kind of molecules that preponderate in the liquid phase, whereas on slow evaporation there has been a possibility for the conversion of this kind of molecules into another, of which the gas phase chiefly consists in ordinary circumstances.

One kind of molecules, which chiefly occurs in liquid ammonia, would then be the inactive kind, and the other kind of molecules, of which the ordinary ammonia gas chiefly consists, the active one. They further pointed out that the existence of inactive and active kinds of molecules probably accounts for the chemical inactivity of the gas dried by BAKER.

So we see that in this paper BALY and DUNCAN already express the supposition at which I had also arrived, though I did not publish it because my investigation was not yet sufficiently advanced. BALY-DUNCAN's results, however, are not very convincing, as BRISCOE¹⁾ observed, because they can also be explained in another way. He says: "It is known, that ordinary commercial ammonia, dried over lime, contains about 1 per cent of water²⁾, and that rapid, irreversible distillation, such as may occur by free discharge of gas from a cylinder of liquid, is a very effective means of separating the constituents even of a constant boiling mixture³⁾, so that the gas thus obtained may well be considerably drier than that in real equilibrium with the cylinder liquid. BALY has found that the addition of water vapour to ordinary ammonia increases its reactivity, drying certainly decreases its reactivity, and so the greater dryness of the "inactive" form would appear, to be capable of explaining the whole of the observations, including the "recovery" of the gas in cylinders on standing (by acquisition of the equilibrium content of water vapour) identity of slowly released cylinder gas with laboratory preparations dried by lime, recovery of inactive gas in the experimental tube, when the wire is heated at 200° (release of absorbed water from the wire or walls) and the increase in reactivity of "inactive" ammonia with increase of temperature of the wire".

These remarks of BRISCOE's, which are very true in my opinion, deprive BALY's published experiments for the present of all their

¹⁾ Annual Reports of the Progress of Chemistry vol. 19 1922, p. 37.

²⁾ Briscoe refers here to WHITE T. 121, 1688 (1922), but this must be a mistake for WHITE has not found this.

³⁾ MULLIKEN J. Amer. Chem. Soc. 44, 2389 (1922).

cogency as a proof of the existence of an active and an inactive kind of molecules in ammonia.

I wanted to test my supposition in another way and took, accordingly, an entirely different course.

After having convinced myself that the pure P_2O_5 which I prepared by BAKER's method, had really the same properties as that of BAKER¹⁾, I began with some of my pupils an investigation of the influence of intensive drying on the point of transition, the melting-point, the vapour tension of the solid and liquid state, and the electrical resistance of the liquid phase of a great number of substances, and among them those substances, of which BAKER found that the chemical activity disappeared by intensive drying, occupy a very particular place on account of the great importance of this phenomenon. Of this latter group first of all NH_3 , HCl , CO , and O_2 were taken in hand.

In a following communication our results and the particulars of the experiments will be discussed.

*Laboratory of General and Inorg.
Chemistry of the University.*

Amsterdam, March 20th 1923.

¹⁾ I became acquainted with this method through a private communication by Prof BAKER before it was published, which saved me a great deal of trouble and time. I will avail myself of this opportunity to express my cordial thanks to Prof. BAKER for his kindness.

Chemistry. — “*The System Sulphur Trioxide*” I. By Prof. A. SMITS.
(Communicated by Prof. P. ZEEMAN).

(Communicated at the meeting of March 24, 1923).

For some years the examination of sulphur trioxide has been on my programme, because I surmised that this substance would yield suitable material to test the theory of allotropy. As, however, other investigations had to go first, this examination could not be taken in hand until a short time ago.

In the meantime BERTHOUD¹⁾ LE BLANC with RÜTLE²⁾ published each a treatise on vapour tensions and melting-points of this substance. Though these two papers will be discussed more at length later on, I will make here already a few remarks, and more particularly in connection with the latter publication.

The results published there prove with the greatest clearness that SO_2 is really a substance which not only can be used as a test of the above-mentioned theory, but which is so eminently fit for it that in this respect it is unequalled by any other. For the results obtained show that both the liquid and the solid phases of the SO_2 can behave as phases of more than one component, which without any doubt must be attributed to the complexity of this phase.

This complexity is owing to the occurrence of different kinds of molecules in the same phase, which molecular-species are in internal equilibrium with each other in the case of unary behaviour. I emphatically pointed out on an earlier occasion that the term “different kinds of molecules” should be taken in as *wide a sense* as possible³⁾. By them we should understand not only the isomer and the polymer molecules, but also the electrically charged dissociation products, ions + electrons, and it is self-evident, that in many cases the difference between molecular-species mentioned here lies in a difference between the atoms. It is in particular the more recent views on the atomic structure, that bring to light, that there are very subtle differences possible between the different atoms, which e.g. are in connection with a change of the quanta-values of the valency-electron-paths, and this leads to kinds of molecules with

¹⁾ Helvetica Chem. Acta 5, 513 (1922).

²⁾ Ber. d. Sächs. Akad. v. Wiss. Leipzig 74, 106 (1922).

³⁾ The theory of Allotropy p. 2.

more *subtle* differences than those, which are assumed between the ordinary isomers. Nevertheless when the idea of "isomery" is taken in a wider sense, also these different kinds of molecules may be classed under this category.

We cannot say as yet what kinds of molecules occur in the different phases of the pure SO_3 . The molecular size in the vapour phase agrees about with SO_3 , but it is very well possible that there occur isomer molecules of SO_3 at the same time, and it is also possible that there is also a polymer kind of molecules present in small concentration. The kinds of molecules that occur in the gas phase, will also be present in the liquid phase, hence according to the theory of allotropy also in the solid phase, though in a different proportion, when the idea molecular of conception is taken in a wide sense ¹⁾. Up to now we have been completely in the dark as far as the internal state of solid SO_3 is concerned. The measurements of the surface tension can, indeed, extend our knowledge concerning the complexity of the liquid phase somewhat, but we still lack means to decide whether a unary solid phase is a mixed crystal in internal equilibrium or not.

Contrary to LE BLANC's opinion it is not possible to conclude to the molecular size of a substance in the solid state in a solvent from the found mol. weight of this substance. ²⁾

With a view to supplementing our methods of research with those that make use of RÖNTGEN rays in the hope of learning something more in the end about the more delicate inner state of equilibrium in the solid phase, I instituted a department for the RÖNTGEN investigation of the solid substance in my laboratory some years ago. Though the way which I had decided to follow, leads to the typical allotropic substances, it seemed desirable first to examine some simple, but nevertheless very interesting, substances, in which results were to be expected which might be of great importance for getting a clearer insight into the nature of the chemical bond. Accordingly Messrs J. M. BIJVOET and A. KARSEN have studied Li, LiH, NaClO_3 , NaBrO_3 , in which it was possible to determine the structure and the binding of the particles on definite suppositions. ³⁾ Now the investigation of HgI_2 has been taken in hand, though we know that by means of this investigation we shall not be able to decide whether the solid phase is a mixed crystal.

¹⁾ Cf. "The Theory of Allotropy" p. 220.

²⁾ Loc. cit.

³⁾ Partly published in These Proc. **23**, 644, 1365 (1921); **25**, 27 (1922); Zeitschr. f. Physik. **14**, 291 (1923).

The investigation by means of RÖNTGEN rays is by no means so powerful as it is often supposed to be. Thanks to the researches of BAKHUIS ROOZEBOOM and his pupils we have got to understand the behaviour of the mixed crystal phases in binary systems to a great extent, but what does the RÖNTGEN investigation teach us about these mixed crystals?

Let us e.g. take the simple system KCl, KBr , a system of which we know that the solid components are homogeneously mixable in all proportions, and let us now suppose an arbitrary mixed crystal from this continuous series to be given to a RÖNTGEN analyst. If this investigator is under the impression that he has to do with a solid phase of a simple substance, he will interpret the intensities found in the usual way, and will find them in very good agreement with the image of the system that was supposed by him to be *mono-componential*. For the intensities can only serve as a test of an already assumed model, and as there are still so many factors that are not sufficiently accurately known in the interpretation of these intensities, and because besides there are nearly always some parameters that have to be chosen so as to suit, a good agreement can be found, even when the supposition is erroneous.

Partly in consequence of these circumstances, partly in consequence of the impossibility to give already now a sharp image of the complexity, as this has also been assumed by me for the solid phase, the RÖNTGEN investigation, in its present stage of development, cannot serve as yet for a further elaboration of the theory of allotropy, and it will, no doubt, be still some years before the RÖNTGEN research will be able to throw new light on the inner equilibria, which have already been found in the solid state.

All the same we have started the RÖNTGEN study of the interesting HgI_2 , because we wished in any case to ascertain if any changes occur in the RÖNTGEN spectrum of these compounds in the temperature interval of $130-255^\circ$, and, if so, what changes, hoping that some conclusions may be drawn from this with some probability.

I have thought it necessary to publish the above discussion, because a great many mistaken ideas still prevail in this region.

When we now return to LE BLANC's investigation, I will remark that he found, among other things, that on cooling of the supercooled liquid below 13.9° solidification suddenly sets in, on which the vapour tension appeared to have risen, also after the temporary rise of temperature had disappeared. Hence at the same temperature the solid phase formed presented a higher pressure than the supercooled liquid, and LE BLANC thought this phenomenon comparable

with the action of oxygen on phosphorus, in which ozone and a phosphoro-oxygen compound was formed.

This, statement shows very clearly the insuperable difficulty with which one is confronted, when with phenomena which so clearly point to the complex character of the phases, one yet continues to occupy the old standpoint.

I will not treat the phenomena found in the examination of SO_2 more at length here, but leave the discussion of them to the following communication.

Amsterdam, March 1923.

*Laboratory of General and Inorganic
Chemistry of the University.*

Geology. -- "*Geological data derived from the region of the
"Bird's head" of New-Guinea*". By Prof. L. RUTTEN.

(Communicated at the meeting of March 24, 1923).

The great northwestern Peninsula of New-Guinea is one of the least known parts of the Indian Archipelago. In recent times some data concerning it have been published by R. D. M. VERBEEK in his "*Molukken Verslag*"¹⁾, and C. E. A. WICHMANN, when journeying from the east coast to Horna, discovered a folded coal-bearing formation²⁾ which proved to be of tertiary age³⁾.

In the last few years (between 1917 and 1921), however, explorations were made on a large scale in Northern New-Guinea and also in the "Bird's head" for oil and coal, by the officers of the Mining Department in the Dutch East Indies. The results of these explorations have not been published as yet⁴⁾, but some years ago I received from the Director of the Mining Department in the Dutch East-Indies a rather large collection of limestones and marls for examination. The study of this collection has been finished, but there would be little sense in expatiating on it here, a fortiori as a description will probably be published elsewhere. It may be of interest though, to summarize the obtained results.

Although we are not quite sure that all the rocks we examined, are of tertiary age, this may yet be assumed for the great majority. Now, when observing on the subjoined sketch-map the localities of "Bird's head" from which the examined rocks are derived, we realize at once that *tertiary deposits have a wide distribution in the north-west part of New-Guinea*. However, *eocene rocks seem to be scarce* among the tertiary deposits, which is quite in keeping with what we know about the other parts of New-Guinea. They were found only in two regions: in the first place between the island of Rumberpon and Horna, where, in two localities, Nummulites-Alveolina limestone and Alveolina-Lacazina limestone have been

¹⁾ Jaarboek Mijnwezen Ned. Indië 1908. Wetensch. Gedeelte.

²⁾ Nova Guinea. IV. 1917.

³⁾ Nova Guinea. VI. 2. 1914.

⁴⁾ I. C. O.-Commissie, The history and present state of scientific research in the Dutch East Indies. Geology. p. 28. 1923.

collected, as well as oligomiocene limestones; while Lacazina-limestones have been found near the Campong Horna; in the second



Eocene	= Eocene.
Oud Neogeen	= Older Neogene.
Jong Neogeen	= Younger Neogene.
Indifferente Gesteenten, meest tertiair	= Indifferent Rocks, mostly tertiary.
Marine Arkose	= Marine Arcose.
Marmer	= Marble.
Oude Schist-materiaal in Tertiair	= Old Schist-material in Tertiary.

place in the northwestern part of the "Bird's head", where Lacazina-limestones have been collected, at one locality. From this it is evident that eocene is only sparingly distributed; moreover it should be observed that the rocks of the two localities, where Lacazina alone is found, cannot on that account be referred to the eocene with absolute certainty, however probable this may be. From the region between Rumberpon (Amberpon) and Horna rocks have been described by me formerly that pointed to the boundary strata between eogene and neogene¹).

On the contrary *limestones of littoral facies from the older neogene* have been found in a large number of localities, characterized by the occurrence of *Lepidocyclina*, *Miogypsina* and *Cycloclypeus*. Similar limestones from the region between Rumberpon and Horna and from the Andai-river near Menokwari, have been previously described. They now appear to occur to the west of Rumberpon in a broad zone, running north-south, and to extend farther south

¹) Nova Guinea. VI. 2. 1914.

than Andai, while they can be recognized in a zone running all along the north coast of "the Bird's head" as far as the island of Batanta. It will be seen at a glance that we have to do here with a comparatively narrow zone of older-neogene, which follows the east coast and the north coast of the "Bird's head". It may be that older-neogene still occurs also in the more western and southern region of "Bird's head", but it is remarkable that among the numerous rocks from those regions that were examined by me, there was not a single one that could positively be referred to the older neogene. We shall see lower down that this is partly due to the facies of the discovered rocks being indifferent, to our having to do either with non-fossiliferous rocks or with rocks that have been deposited in a deeper sea, in which the fossils, so characteristic of the littoral older neogene, cannot be expected to occur. But beyond these also rocks occur repeatedly in the southern part of the "Bird's head", that are of littoral facies, in which e.g. Lithothamnium, Operculina and Amphistegina, the companions of Lepidocyclus in the older neogene etc., occur, but in which the Foraminifera, which are characteristic of the older neogene, are lacking. In such cases we no doubt have to do with *younger neogene which indeed is often borne out by the habitus of the rocks*. As an instance we point to the basin of the Aer Beraur and of the Aer Klasaman, in which a series of rocks occur that are referable to the younger neogene. Another region of probably young-neogene rocks, partly with true littoral habitus, is situated North of lake Amaru. Between lake Amaru and the Aer Beraur a number of rocks have been found: globigerina marls, fine grained lime sandstones and the like, which are completely indifferent, so that nothing can be said about their age. The same applies to some rocks from the region south of lake Amaru. A long list of rock samples, collected in a west-east zone far north of lake Amaru, are undoubtedly referable to the neogene, but their fossils and their facies are not typical enough to say whether they belong to the older or to the younger neogene. In some rocks, however, doubtful Lepidocyclus were recognized; the others have been classed under the "indifferent rocks". Lastly among the rocks from the basin of the Aer Sebjar there are some littoral limestones, in which no "older" forms are to be found, so that here also we have probably to do with younger neogene. On the other hand, a number of very fine grained lime sandstones and globigerina limes, collected east of Muturi-river have to be classed under the "indifferent rocks". They may be of older-neogene age, because in the adjacent region towards the east (west of Rumberpon) a few transition rocks were

found among true littoral *Lepidocyclina*-limes and *Globigerina*-limes.

Lastly presumably young-neogene rocks are to be found to the North and West of Menokwari. Here *Globigerina* marls and loose limesands, occur, which indeed do not include typical fossils, but which on account of their quite young habitus are most likely to be reckoned to the younger neogene. This in fact agrees with the circumstance that some limestones in this region are of littoral facies but do not contain *Lepidocyclina*, *Cyclocypus* or *Miogypsina*. Before this a description was published of limestones from the island of Manaswari, near Menakwari, that were considered to be younger-neogene ¹⁾).

Between the localities of old-neogene limes south of Menokwari and those west of Rumberpon are situated the high Arfak Mountains, which according to VERBEEK ²⁾ and WICHMANN ³⁾ are composed of granular eruptive rocks, schists and slates. From the region of the Arfak mountains I received three rocks most likely tertiary and built up of *detritus from the Arfak Mountains*. They are coarse-grained arcoses of marine origin, which together with Corals also contain a very few *Globigerina*. The minerals represented here are much quartz, orthoclase, perthite and less plagioclase and biotite: apparently we have to do here with the detritus of acid granites.

Coarse-grained *detritus of old rocks* occurs also frequently in the northern part of "the Bird's head" in the rocks of tertiary age — notably in the old-neogene rocks. This goes to show that below, and perhaps also at the surface, *there must exist a mountain range of older rocks*. The localities marked on the map by an σ are those where in the limestones transported fragments of quartzite and phyllite occur. A rock from the basin of the Aer Sebjar contained grains of perthite and orthoclase, which remind us of the detritus rocks of the Arfak mountains.

The future reports of the Mining Department will undoubtedly contain interesting information on these "older rocks" in the "Bird's head".

¹⁾ Nova Guinea. VI. 2. p. 29. 42.

²⁾ Nova Guinea IV. p. 97.

³⁾ Tijdschr. Kon. Ned. Aardr. Gen. (2). 21. 1904.

Mathematics. — “A theorem concerning power-series in an infinite number of variables, with an application to DIRICHLET’S ¹⁾ series.” By H. D. KLOOSTERMAN. (Communicated by Prof. J. C. KLUYVER.)

(Communicated at the meeting of March 24, 1923).

§ 1. An important relation between the theory of DIRICHLET’S series and the theory of power-series in an infinite number of variables (for abbreviation we shall write: power-series in an i. n. of v.) has been discovered by H. BOHR ²⁾. Let

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad s = \sigma + it \dots \dots \dots (1)$$

be an ordinary DIRICHLET’S series. Put $x_1 = \frac{1}{2^s}$, $x_2 = \frac{1}{3^s}$, $\dots \dots x_m = \frac{1}{p_m^s}$, \dots (where p_m is the m -th prime-number, and let $n = p_{n_1}^{v_1} p_{n_2}^{v_2} \dots p_{n_r}^{v_r}$, where $p_{n_1}, p_{n_2}, \dots p_{n_r}$ are the different primes which divide n . Then the series (1) can *formally* be written as a power-series in an i. n. of v., thus:

$$P(x_1, x_2, \dots x_m, \dots) = \sum_{n=1}^{\infty} a_n x_{n_1}^{v_1} x_{n_2}^{v_2} \dots x_{n_r}^{v_r} =$$

$$c + \sum_{\alpha=1,2,\dots} c_{\alpha} x_{\alpha} + \sum_{\substack{\alpha, \beta=1,2,\dots \\ \alpha \leq \beta}} c_{\alpha, \beta} x_{\alpha} x_{\beta} + \sum_{\substack{\alpha, \beta, \gamma=1,2,\dots \\ \alpha \leq \beta \leq \gamma}} c_{\alpha, \beta, \gamma} x_{\alpha} x_{\beta} x_{\gamma} + \dots$$

This relation has been applied by BOHR to the so-called absolute-convergence-problem for DIRICHLET’S series, that is to say the determination of the abscissa of absolute convergence of (1) (the lower bound of all numbers β , such that the series (1) converges for $\sigma \geq \beta$, in terms of (preferably as simple as possible) analytic properties of the function represented by (1). Let B be the abscissa of absolute convergence of (1), and D the lower limit of all numbers α , such that $f(s)$ is *regular* and *bounded* for $\sigma \geq \alpha$. The absolute-convergence-problem will be solved, if the difference $B - D$ is known. BOHR proves that $B = D$ for any DIRICHLET’S series that can be *formally* represented in one of the following forms:

¹⁾ A more detailed proof of the theorem will be published elsewhere.

²⁾ Göttinger Nachrichten, 1918.

$$f(s) = \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \frac{a_m^l}{(p_m^l)^s}$$

or

$$f(s) = \prod_{m=1}^{\infty} \left(1 + \sum_{l=1}^{\infty} \frac{a_m^l}{(p_m^l)^s} \right),$$

or, what comes to the same thing, for any DIRICHLET's series for which the connected power-series in an i. n. of v . has one of the forms

$$P(x_1, x_2, \dots, x_m, \dots) = \sum_{n=1}^{\infty} Q_n(x_n) \quad . \quad . \quad . \quad (2)$$

or

$$P(x_1, x_2, \dots, x_m, \dots) = \prod_{n=1}^{\infty} (1 + Q_n(x_n)) \quad . \quad . \quad . \quad (3)$$

where $Q_n(x_n)$ ($n = 1, 2, \dots$) is a power-series in x_n without a constant term. The equality $B = D$ is a consequence of the theorem:

If: a. The series is *bounded*¹⁾ for $|x_n| \leq G_n$ ($n = 1, 2, \dots$), then
 b. it is absolutely convergent for $|x_n| \leq \theta G_n$, where θ is an arbitrary positive number in the interval $0 < \theta < 1$).

Now, if we consider the power-series (2) and (3), we see that the variables x_n occur to some extent separated from one another. This led BOHR to the conjecture, that the equality $B = D$ would hold for any DIRICHLET's series, for which the variables in the connected power-series in an i. n. of v . do not occur too much mixed up. Confirmation of this conjecture is the purpose of the present com-

¹⁾ According to HILBERT (Wesen und Ziele einer Analysis der unendlich vielen unabhängigen Variablen, Palermo Rendiconti, vol. 27, p. 67) a power-series in an i. n. of v . is defined to be *bounded* if:

1^o. The power-series $P_m(x_1, x_2, \dots, x_m)$ (*Abschnitte*), that may be obtained from the power-series in an i. n. of v . by putting $x_{m+1} = x_{m+2} = \dots = 0$, are, for all values of m , absolutely convergent in the region $|x_1| \leq G_1, |x_2| \leq G_2, \dots, |x_m| \leq G_m$.

2^o. There exists a number K , independent of m , such that, for every m , the inequality

$$|P_m(x_1, x_2, \dots, x_m)| < K$$

holds in the region $|x_1| \leq G_1, |x_2| \leq G_2, \dots, |x_m| \leq G_m$.

²⁾ It is well known, that b follows from a for any power-series in a *finite* number of variables. Originally HILBERT had assumed this also, as being self-evident, for an i. n. of v . But BOHR showed that this could not be true by constructing an example to the contrary.

munication. In fact it can be proved that $B = D$ holds for any DIRICHLET'S series that can *formally* be written in the form

$$f(s) = \varphi \cdot \left(\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \frac{a_{p_m^l}}{(p_m^l)^s} \right),$$

where φ is an arbitrary (non-constant)¹⁾ integral function. As a consequence of the relation, already mentioned above several times, the following theorem concerning power-series in an i. n. of v. is equivalent to this statement.

Theorem. If φ is an integral function and $Q_n(x_n)$ ($n=1, 2, \dots$) a *formal*²⁾ power-series in x_n , without a constant term, and if the power-series in an i. n. of v. $P(x_1, x_2, \dots, x_m, \dots) = \varphi(Q_1(x_1) + Q_2(x_2) + \dots + Q_m(x_m) + \dots)$ is bounded for $|x_n| \leq G_n$ ($n=1, 2, \dots$), then it is absolutely convergent for $|x_n| \leq \theta G_n$, if $0 < \theta < 1$.

In the following pages an outline of the proof of this theorem will be given.

§ 2. For the sake of simplicity we take $G_1 = G_2 = \dots = G_n = G > 1$, but $\theta G < 1$.

Because the given power-series in an i. n. of v. is bounded, there exists a number K , not depending on m , such that

$$|\varphi(Q_1(x_1) + Q_2(x_2) + \dots + Q_m(x_m))| < K. \quad (4)$$

The first part of the proof of the theorem of § 1 discusses the power-series $Q_n(x_n)$ ($n=1, 2, \dots$). It is proved that it follows from (4) that all these power-series possess a certain region of convergence. Further research shows that two cases may occur:

1°. The functions $Q_n(x_n)$ are all regular for $|x_n| < G$. This is the general case.

2°. If the integral function $\varphi(y)$ has the form $V\left(e^{\frac{y}{M}}\right)$ (where V is again an integral function), then it is only possible to conclude that the functions $Q_n(x_n)$ are logarithms of functions regular for $|x_n| < G$, namely that they have the form $Q_n(x_n) = \log(1 + R_n(x_n))$, where $R_n(x_n)$ is regular for $|x_n| < G$, and $R_n(0) = 0$ ³⁾.

¹⁾ If φ is a constant, the theorem is trivial.

²⁾ That is to say, the existence of a region of convergence is not assumed, but will appear to be a consequence of the other assumptions.

³⁾ It is interesting to observe, that obviously the series (2), with $\varphi(y) = y$, falls under the first case, and the series (3), with $\varphi(y) = e^y$, $V(z) = z$, under the second case.

For shortness' sake we confine ourselves to the first case. (The proof in the second case is not essentially different, though in details more intricate). Then the functions $Q_n(x_n)$ are, because $G > 1$, all regular in their resp. circles $|x_n| \leq 1$.

For any function $f(z)$, regular for $|z| \leq 1$, and for which $f(0) = 0$, we now define a number r as follows: r is the radius of the largest circle, of which all points represent numbers assumed by $f(z)$ in the circle $|z| \leq 1$. Let r_n ($n = 1, 2, \dots$) be the corresponding quantity for $Q_n(x_n)$. Then we first prove, that the series $\sum_{n=1}^{\infty} r_n$ converges.

For this purpose we consider (4), valid for all sets of values of x_1, x_2, \dots, x_m , satisfying $|x_n| \leq G$ ($n = 1, 2, \dots, m$), and, *a fortiori*, for all satisfying $|x_n| \leq 1$. Because $\varphi(y)$ is an integral function, it is possible to choose a number L so large, that the maximum value of $|\varphi(y)|$, on the circle $|y| = L$, is $> K$. Now suppose that, for some value of m , $r_1 + r_2 + \dots + r_m > L$. Then the maximum value of $|\varphi(y)|$ on the circle $|y| = r_1 + r_2 + \dots + r_m$ would be $> K$. Now if we let the variables x_n ($n = 1, 2, \dots, m$) describe their resp. circles $|x_n| \leq 1$, then $Q_n(x_n)$ assumes all values satisfying $|Q_n(x_n)| = r_n$. Hence $y = Q_1(x_1) + Q_2(x_2) + \dots + Q_m(x_m)$ assumes all values satisfying $|y| = r_1 + r_2 + \dots + r_m$. Therefore it would be possible to find a set of values x'_1, x'_2, \dots, x'_m such that

$$y = Q_1(x'_1) + Q_2(x'_2) + \dots + Q_m(x'_m) = (r_1 + r_2 + \dots + r_m)e^{i\psi},$$

where $(r_1 + r_2 + \dots + r_m)e^{i\psi}$ represents that point of the circle $|y| = r_1 + r_2 + \dots + r_m$ where $|\varphi(y)|$ assumes its maximum value. Therefore we should have

$$|\varphi(Q_1(x'_1) + Q_2(x'_2) + \dots + Q_m(x'_m))| > K,$$

contradictory to (4). Therefore the supposition $r_1 + r_2 + \dots + r_m > L$ can not be true. Since L is independent of m , this proves the convergence of $\sum_{n=1}^{\infty} r_n$.

We now apply the following theorem of BOHR¹⁾:

Let the function $f(z) = \sum_{n=1}^{\infty} a_n z^n$ ($f(0) = 0$) be regular for $|z| \leq 1$.

Let $M(\varrho)$ be the maximum value of $|f(z)|$ on the circle $|z| = \varrho$ ($0 < \varrho < 1$). Then, if r is the quantity defined above, we have $r \geq k M(\varrho)$, where k is a number which depends on ϱ only (k is

¹⁾ Not yet published.

therefore the same for all functions satisfying the assumptions of the theorem).

Hence, if $M_n(\varrho)$ is the maximum value of $|Q_n(x_n)|$ on the circle $|x_n| = \varrho$ ($n = 1, 2, \dots$), we have $r_n \geq k M_n(\varrho)$. Since we have proved that $\sum_{n=1}^{\infty} r_n$ is convergent, it now follows that the series $\sum_{n=1}^{\infty} M_n(\varrho)$ converges also (for $\varrho < 1$). From this fact the theorem of § 1 can be easily deduced.

For let $Q_n(x_n) = \sum_{p=1}^{\infty} a_p^{(n)} x_n^p$ ($n = 1, 2, \dots$). Then

$$|a_p^{(n)}| \leq \frac{M_n(\varrho)}{\varrho^p} \begin{pmatrix} n = 1, 2, \dots \\ p = 1, 2, \dots \end{pmatrix} (\varrho < 1).$$

If $\Theta = \theta G$ (where θ is the constant of § 1), then it follows that, if $\Theta < \varrho < 1$, (we take for example $\varrho = \frac{1 + \Theta}{2}$),

$$\sum_{p=1}^{\infty} |a_p^{(n)}| \Theta^p \leq \frac{2 \Theta M_n(\varrho)}{1 - \Theta}.$$

Hence the series

$$\sum_{n=1}^{\infty} \sum_{p=1}^{\infty} |a_p^{(n)}| \Theta^p,$$

is also convergent. This proves a *fortiori* the convergence of the given power-series in an i. n. of v. for $|x_n| \leq \Theta = \theta G$ ($n = 1, 2, \dots$).

It cannot be denied that the assumption, that q is an integral function, is somewhat unaesthetic. However, the author has not succeeded in dealing with the more general problem, where φ is an arbitrary (purely formal) power-series. In any case the method described does not give the required result in the more general case.

Copenhagen, November 1922.

Chemistry. — “*In-, mono- and divariant equilibria.*” XXIII. By
Prof. F. A. H. SCHREINEMAKERS.

(Communicated at the meeting of March 24, 1923).

Equilibria of n components in $n + 1$ phases, when the quantity of one of the components approaches to zero. The influence of a new substance on an invariant equilibrium. (Continuation).

We write the isovolumetrical reaction of an equilibrium $E(x=0)$:

$$\lambda_1 F_1 + \lambda_2 F_2 + \dots = 0 \quad \Sigma (\lambda H)_V > 0 \quad \Sigma (\lambda V) = 0. \quad (1)$$

and the isentropical reaction:

$$\mu_1 F_1 + \mu_2 F_2 + \dots = 0 \quad \Sigma (\mu H) = 0 \quad \Sigma (\mu V)_H > 0. \quad (2)$$

Consequently in reaction (1) are formed on addition of heat and in reaction (2) on increase of volume those phases, which have a negative reaction-coefficient. We have, therefore:

$$\Sigma (\lambda x)_V = -\lambda_1 x_1 - \lambda_2 x_2 - \dots \quad \Sigma (\mu x)_H = -\mu_1 x_1 - \mu_2 x_2 - \dots$$

When we subtract both reaction-equations (1) and (2) from one another, after having multiplied the first one with μ_1 and the second one with λ_1 , then we find the reaction:

$$(\mu_1 \lambda_2 - \lambda_1 \mu_2) F_2 + (\mu_1 \lambda_3 - \lambda_1 \mu_3) F_3 + \dots = 0 \quad (3)$$

wherein the change of entropy is $\mu_1 \Sigma (\lambda H)_V$

and the change of volume is $-\lambda_1 \Sigma (\mu V)_H$.

As (3) represents the reaction, which may occur in the equilibrium $(F_1) = F_2 + F_3 + \dots$, we have

$$\left(\frac{dP'}{dT} \right)_1 = - \frac{\mu_1}{\lambda_1} \cdot \frac{\Sigma (\lambda H)_V}{\Sigma (\mu V)_H} \quad (4)$$

Herein $\left(\frac{dP'}{dT} \right)_1$ indicates the direction of curve (F_1) in the invariant point. In the same way we find:

$$\left(\frac{dP}{dT} \right)_2 = - \frac{\mu_2}{\lambda_2} \cdot \frac{\Sigma (\lambda H)_V}{\Sigma (\mu V)_H}; \quad \left(\frac{dP}{dT} \right)_3 = - \frac{\mu_3}{\lambda_3} \cdot \frac{\Sigma (\lambda H)_V}{\Sigma (\mu V)_H} \text{ etc.} \quad (5, 6)$$

As we are able to deduce from (1) and (2) also the direction of temperature and pressure of the different monovariant curves, the P, T -diagram is, therefore, quantitatively defined.

Now we add to the equilibrium a new substance X , which occurs

in the phases F, F_1, \dots with the concentrations x_1, x_2, \dots . In accordance with (13) and (15) (XXII) we now have:

$$\frac{\Sigma (\lambda H)_V}{RT} \cdot (dT)_x = \lambda_1 x_1 + \lambda_2 x_2 + \dots = -\Sigma (\lambda x)_V. \quad (7)$$

$$\frac{\Sigma (\mu V)_H}{RT} \cdot (dP)_x = -\mu_1 x_1 - \mu_2 x_2 - \dots = \Sigma (\mu x)_H. \quad (8)$$

With the aid of (4) etc. we may also write for this:

$$\frac{\Sigma (\mu V)_H}{RT} \cdot (dT)_x = -x_1 \mu_1 \left(\frac{dT}{dP} \right)_1 - x_2 \mu_2 \left(\frac{dT}{dP} \right)_2 - \dots \quad (9)$$

$$\frac{\Sigma (\lambda H)_V}{RT} \cdot (dP)_x = x_1 \lambda_1 \left(\frac{dP}{dT} \right)_1 + x_2 \lambda_2 \left(\frac{dP}{dT} \right)_2 + \dots \quad (10)$$

It follows from (8) and (9):

$$\left(\frac{dT}{dP} \right)_x = -\frac{x_1 \mu_1}{\Sigma (\mu x)_H} \cdot \left(\frac{dT}{dP} \right)_1 - \frac{x_2 \mu_2}{\Sigma (\mu x)_H} \cdot \left(\frac{dT}{dP} \right)_2 - \dots \quad (11)$$

from (7) and (10) it follows:

$$\left(\frac{dP}{dT} \right)_x = -\frac{x_1 \lambda_1}{\Sigma (\lambda x)_V} \cdot \left(\frac{dP}{dT} \right)_1 - \frac{x_2 \lambda_2}{\Sigma (\lambda x)_V} \cdot \left(\frac{dP}{dT} \right)_2 - \dots \quad (12)$$

and from (7) and (8):

$$\frac{\Sigma (\mu V)_H}{\Sigma (\lambda H)_V} \cdot \left(\frac{dP}{dT} \right)_x = -\frac{\mu_1 x_1 + \mu_2 x_2 + \dots}{\lambda_1 x_1 + \lambda_2 x_2 + \dots} \quad (13)$$

From (7) we see that we are able to express $(dT)_x$ with the aid of the isovolumetrical reaction (1); it is apparent from (9) that; however, we cannot express $(dT)_x$ with the aid of the isentropical reaction (2) only, but that we must know also the directions of the monovariant curves $(F_1) (F_2) \dots$ of the equilibrium $E(x=0)$.

It appears from (8) that we are able to express $(dP)_x$ with the aid of the isentropical reaction (2); we see, however, from (10) that we cannot define $(dP)_x$ with the aid of the isovolumetrical reaction only but that we must know for this also again the directions of the curves $(F_1) (F_2) \dots$.

The direction of the monovariant curve E can be defined, as is apparent from (13), with the aid of the isovolumetrical and isentropical reaction; it follows from (11) and (12) that it can also be defined with the aid of the directions of the curves $(F_1) (F_2) \dots$ and one of both reactions.

When we add a new substance X which occurs in one of the phases only, f.i. in F_1 than we must put in (7)–(13) $x_3=0, x_3=0 \dots$. As now $\Sigma (\lambda x)_V = -\lambda_1 x_1$, it follows from (12):

$$\left(\frac{dP}{dT} \right)_x = \left(\frac{dP}{dT} \right)_1 \quad \dots \quad (14)$$

which follows of course immediately from (11). Consequently curve E and (F_1) have the same tangent in the invariant point. It follows from (7) and (8) that they go also in the same direction of temperature and pressure, starting from this point. When viz. λ_1 is positive, then it follows from reaction (1) that curve (F_1) goes towards higher temperatures, starting from the invariant point. As it follows, however, from (7) that $(dT)_{x_1}$ is then positive also, consequently curve E goes also towards higher T . When λ_1 is negative, then the curves (F_1) and E go both towards lower T . It follows from (2) and (8) that both curves have also the same direction of pressure.

In accordance with previous papers (Communication XXII) we, therefore, find: when the new substance occurs in the phase F_1 only, then curve E coincides with curve (F_1) .

When the new substance occurs in the phases F_1 and F_2 only, then (12) passes into:

$$\left(\frac{dP}{dT}\right)_x = \frac{\lambda_1}{\lambda_1 + K\lambda_2} \left(\frac{dP}{dT}\right)_1 + \frac{K\lambda_2}{\lambda_1 + K\lambda_2} \left(\frac{dP}{dT}\right)_2 \quad (15)$$

wherein $K = \frac{x_2}{x_1}$. Hence it follows:

$$d\left(\frac{dP}{dT}\right)_x = \frac{\lambda_1\lambda_2}{(\lambda_1 + K\lambda_2)^2} \left[\left(\frac{dP}{dT}\right)_2 - \left(\frac{dP}{dT}\right)_1 \right] dK \quad (16)$$

For fixing the ideas we assume that $\left(\frac{dP}{dT}\right)_2$ is greater than $\left(\frac{dP}{dT}\right)_1$. Now we distinguish two cases.

1. λ_1 and λ_2 have the same sign. The following is apparent from (15) and (16). When K changes from 0 to ∞ then $\left(\frac{dP}{dT}\right)_x$ increases from $\left(\frac{dP}{dT}\right)_1$ to $\left(\frac{dP}{dT}\right)_2$ without becoming maximum, minimum or discontinuous.

2. λ_1 and λ_2 have opposite sign. When K changes from 0 to ∞ , then $\left(\frac{dP}{dT}\right)_x$ decreases without becoming maximum or minimum from $\left(\frac{dP}{dT}\right)_1$ till $-\infty$, then it proceeds discontinuously towards $+\infty$ and afterwards it decreases to $\left(\frac{dP}{dT}\right)_2$.

When λ_1 and λ_2 are both positive, then, in accordance with reaction (1) both curves (F_1) and (F_2) go towards higher tempera-

tures starting from the invariant point; when λ_1 and λ_2 are both negative, then both curves go towards lower T ; when λ_1 and λ_2 have opposite sign, then both curves go, starting from the invariant point in opposite direction of temperature.

It follows from all this that the tangent to curve E is situated within the angle, which is formed by the curves (F_1) and (F_2) . [Of course we mean that angle which is smaller than 180°]. As in the case of $K=0$ (consequently $x_2=0$) curve E coincides with (F_1) and in the case of $K=\infty$ (consequently $x_1=0$) curve E coincides with (F_2) consequently the property follows, which we have deduced already in the previous communication also, viz:

Curve E is situated between the curves (F_1) and (F_2) or in other words: in the region $(F_1 F_2)$.

Yet also we find, however:

Curve E is situated nearer curve (F_1) in proportion as the concentration of the new substance in the phase F_1 is larger with respect to that in F_2 ; curve E is situated nearer to curve (F_2) in proportion as the concentration of the new substance in the phase F_2 is greater with respect to that in F_1 .

When the new substance occurs only in the phases $F_1 F_2$ and F_3 , then we find, in accordance with previous papers that curve E is situated in the region $(F_1 F_2 F_3)$.

When one of the curves, f.i. (F_3) is between the other two (F_1) and (F_2) then curve E is situated also between (F_1) and (F_2) . When, however, none of the three curves is situated between the other two, then curve E may go, starting from the invariant point in every arbitrary direction.

Now we consider the binary equilibrium

$$E(x=0) = F + L_1 + L_2 + G$$

we represent the composition, the entropy and the volume of

F by y $1-y$ H and V

L_1 „ y_1 $1-y_1$ H_1 and V_1

L_2 „ y_2 $1-y_2$ H_2 and V_2

G „ y_3 $1-y_3$ H_3 and V_3

When we add a new substance X , then we call its concentration in those phases x x_1 x_2 and x_3 .

In order to deduce the isovolumetrical and isentropical reaction we take two arbitrary reactions; for this we choose:

$$F + a L_2 \rightleftharpoons (1 + a) L_1 \quad \Delta H \Delta V \quad . \quad . \quad . \quad (17)$$

$$(1 + b) L_2 \rightleftharpoons F + b G \quad \Delta H' \Delta V' \quad . \quad . \quad . \quad (18)$$

Herein is:

$$\begin{aligned} \Delta H &= (1 + a) H_1 - H - a H_2, & \Delta H' &= H + b H_2 - (1 + b) H_1, \\ \Delta V &= (1 + a) V_1 - V - a V_2, & \Delta V' &= V + b V_2 - (1 + b) V_1. \end{aligned}$$

In (17) and (18) a and b may be as well positive as negative. It follows from (17) and (18) for the isovolumetrical reaction:

$$\begin{aligned} (\Delta V + \Delta V') F - (1 + a) \Delta V' L_1 + [a \Delta V' - (1 + b) \Delta V] L_2 + b \Delta V G = 0 \\ \Delta H \Delta V' - \Delta H' \Delta V = 0 \quad . \quad . \quad . \quad (19) \end{aligned}$$

and for the isentropical reaction:

$$\begin{aligned} -(\Delta H + \Delta H') F + (1 + a) \Delta H' L_1 - [a \Delta H' - (1 + b) \Delta H] L_2 - b \Delta H G = 0 \\ 0 \quad \Delta H \Delta V' - \Delta H' \Delta V \quad . \quad . \quad . \quad (20) \end{aligned}$$

We now add to this equilibrium $E(x=0)$ a new substance X_2 which occurs in the two liquids L_1 and L_2 only. With the aid of (19) and (20) it then follows from (7) and (8):

$$M \cdot (dT)_x = - (1 + a) \Delta V' x_1 + [a \Delta V' - (1 + b) \Delta V] x_2 \quad (21)$$

$$M \cdot (dP)_x = - (1 + a) \Delta H' x_1 + [a \Delta H' - (1 + b) \Delta H] x_2 \quad (22)$$

wherein:

$$M = (\Delta H \Delta V' - \Delta H' \Delta V) : R T$$

It follows from (21) and (22): when we add to the equilibrium $E(x=0)$ a new substance which occurs only in the two liquids, then the temperature as well as the pressure may be increased or decreased.

We now shall assume that the four phases are situated with respect to one another, as on the line YZ in fig. 1. Then we have:

$$y > y_1 > y_2 > y_3.$$

It follows from (17) and (18) for the determination of a and b :

$$\begin{aligned} y + a y_2 &= (1 + a) y_1 & (1 + b) y_2 &= y + b y_3 \\ a &= \frac{y - y_1}{y_1 - y_2} & b &= \frac{y - y_2}{y_2 - y_3} \quad . \quad . \quad . \quad (23) \end{aligned}$$

so that a and b are positive. Further we assume that F and L_1 and also that L_1 and L_2 are not situated very close to one another, so that a is neither very small nor very large. When F and L_2 and also L_2 and G are not situated very close to one another, then also b is not very small and not very large.

As now $\Delta V'$ is positive and very large with respect to ΔV , M is positive.

Further we may distinguish the following cases.

$$a) \quad \left. \begin{array}{ll} \Delta H > 0 & \Delta V \geq 0 \\ a \Delta H' - (1+b) \Delta H > 0 \end{array} \right\} \quad \left. \begin{array}{ll} \Delta H' > 0 & \Delta V' > 0 \\ a \Delta V' - (1+b) \Delta V > 0 \end{array} \right\} \quad (24)$$

$$b) \quad \left. \begin{array}{ll} \Delta H > 0 & \Delta V \geq 0 \\ a \Delta H' - (1+b) \Delta H < 0 \end{array} \right\} \quad \left. \begin{array}{ll} \Delta H' > 0 & \Delta V' > 0 \\ a \Delta V' - (1+b) \Delta V > 0 \end{array} \right\} \quad (25)$$

$$c) \quad \left. \begin{array}{ll} \Delta H > 0 & \Delta V \geq 0 \\ a \Delta H' - (1+b) \Delta H < 0 \end{array} \right\} \quad \left. \begin{array}{ll} \Delta H' < 0 & \Delta V' > 0 \\ a \Delta V' - (1+b) \Delta V > 0 \end{array} \right\} \quad (26)$$

In each of the three cases, mentioned above, is in (21) the coefficient of x_1 negative and of x_2 positive; consequently we have:

$$(dT)_x \geq 0 \text{ when } \frac{x_2}{x_1} \geq \frac{(1+a) \Delta V'}{a \Delta V' - (1+b) \Delta V} \quad \dots \quad (27)$$

As $\Delta V'$ is very large with respect to ΔV it follows from this approximately with the aid of (23):

$$(dT)_x \geq 0 \text{ when } \frac{x_2}{x_1} \geq \frac{y-y_1}{y-y_2} \quad \dots \quad (28)$$

In the case, mentioned sub b in (22) the coefficients of x_1 and x_2 are negative, so that $(dP)_x$ is also negative; consequently the pressure is lowered.

In order to examine more in detail the sign of $(dP)_x$ we write for (22)

$$M (dP)_x = \left[x_2 - \frac{\Delta H'}{\Delta H' - \frac{1+b}{a} \Delta H} \cdot \frac{1+a}{a} x_1 \right] N \quad \dots \quad (29)$$

wherein:

$$N = a \Delta H' - (1+b) \Delta H$$

When we put herein the value of a from (23) then we may write for (29):

$$M \cdot (dP)_x = \left[\frac{x_2}{x_1} - \frac{\Delta H'}{\Delta H' - \frac{1+b}{a} \Delta H} \cdot \frac{y-y_2}{y-y_1} \right] N x_1 \quad \dots \quad (30)$$

When we consider the three cases a , b and c mentioned above, then we may write for (30):

$$a) \quad (dP)_x = \left[\frac{x_2}{x_1} - (1+K) \frac{y-y_2}{y-y_1} \right] L \quad \dots \quad (31)$$

$$b) \quad (dP)_x = - \left[\frac{x_2}{x_1} + K \frac{y-y_2}{y-y_1} \right] L \quad \dots \quad (32)$$

$$c) \quad (dP)_x = \left[- \frac{x_2}{x_1} + (1-K) \frac{y-y_2}{y-y_1} \right] L \quad \dots \quad (33)$$

wherein L , K , $1 + K$ and $1 - K$ are positive. In each of the three formula's L and K have different values.

In order to apply the above we take the figs. 1 and 2, wherein XP is a side of the components-triangle XYZ . The points F , L_1 , L_2 and G represent the four phases of the invariant binary equilibrium $E(x=0) = F + L_1 + L_2 + G$. When we add a new substance X then the ternary equilibrium $E = F + L_1 + L_2 + G$ arises. The liquids L_1 and L_2 then proceed along the curves $L_1 q_1 r_1$ and $L_2 q_2 r_2$; as the new substance is not volatile, G follows a part of the line XZ . When we add only a little of the new substance, then the liquids are represented by the points q_1 and q_2 in the immediate

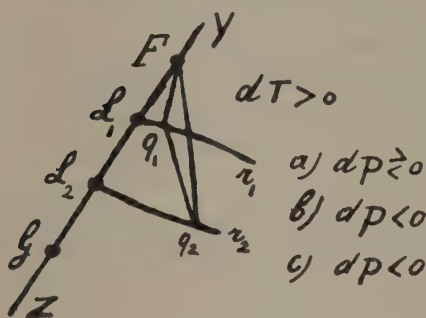


Fig. 1.

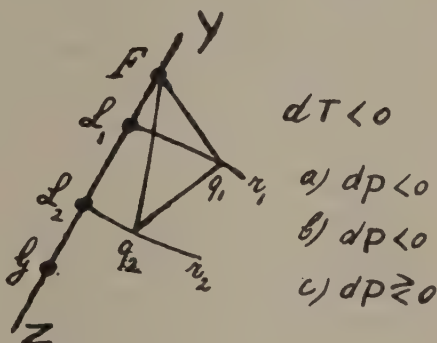


Fig. 2.

vicinity of L_1 and L_2 ; for the sake of clearness they have been drawn in the figures on greater distance.

In fig. 1 is:

$$\frac{x_2}{y-y_2} > \frac{x_1}{y-y_1} \quad \text{or} \quad \frac{x_2}{x_1} > \frac{y-y_2}{y-y_1} \quad . \quad . \quad . \quad (34)$$

consequently in accordance with (28): $(dT)_x > 0$ as is also indicated in the figure. It follows from (31)—(33):

$$\text{in case } a \text{ is } (dP)_x \geq 0$$

$$,, \quad ,, \quad b \quad ,, \quad (dP)_x < 0$$

$$,, \quad ,, \quad c \quad ,, \quad (dP)_x < 0$$

as is also indicated in fig. 1.

In fig. 2 is:

$$\frac{x_2}{y-y_2} < \frac{x_1}{y-y_1} \quad \text{or} \quad \frac{x_2}{x_1} < \frac{y-y_2}{y-y_1} \quad . \quad . \quad . \quad (35)$$

It follows from (28): $(dT)_x < 0$. From (31)—(33) it follows:

$$\begin{array}{ll}
 \text{in case } a \text{ is } (dP)_x < 0 \\
 \text{,, ,, } b \text{ ,, } (dP)_x < 0 \\
 \text{,, / ,, } c \text{ ,, } (dP)_x \geq 0
 \end{array}$$

as is indicated also in fig. 2.

In fig. 1 the pressure may as well increase as decrease in the case *a*; it is apparent from (31) that $(dP)_x$ shall be positive for large values of $x_2 : x_1$. As L_1 (and consequently also q_1) is the liquid which contains the most of the solid substance F we shall call L_1 (and consequently also q_1) the concentrated and L_2 the diluted solution.

We, therefore, find the following:

when the threephases-triangle solid-liquid-liquid turns its concentrated solution towards the side of the components-triangle (fig. 1) then the temperature increases and the pressure generally decreases; only when the concentration of the new substance in the diluted liquid (consequently x_2) is much larger than in the concentrated liquid consequently x_1), then in the case *a* the pressure may increase also.

In fig. 2 in the case *c* the pressure may as well increase as decrease; it appears from (33) that $(dP)_x$ shall be positive for small values of $x_2 : x_1$.

Consequently we find the following:

when the threephases-triangle solid-liquid-liquid turns its concentrated solution away from the side of the components-triangle (fig. 2) then the temperature decreases and generally the pressure also.

Only when the concentration of the new substance is much larger in the concentrated solution (x_1) than in the diluted solution (x_2), then in the case *c* the pressure may also increase.

We may obtain the previous results also by using the P, T -diagram of the equilibrium $E(x=0)$. We may deduce this in the following way.

The direction of temperature of the equilibrium $(G) = F + L_1 + L_2$ is defined by the sign of the coefficient of the phase G in the isovolumetrical reaction (19). As $b \Delta V$ may be as well positive as negative, curve (G) may go, starting from the invariant point i , as well towards higher as towards lower temperatures.

The direction of pressure of the equilibrium (G) is defined by the sign of the coefficient of G in the isentropical reaction (20). As $-b \Delta H$ is negative in each of the cases *a*, *b* and *c*, curve (G) proceeds, starting from the invariant point i , towards higher pressures.

As further, in accordance with (17):

$$\left(\frac{dP}{dT}\right)_G = \frac{\Delta H}{\Delta V}$$

and ΔV is very small, curve (G) is ascending, starting from point i fast vertically. In figs 3 and 4 this curve is drawn vertically upwards; the double arrow indicates that starting from i , it may run either towards the right or to the left.

As the coefficient $-(1+a)\Delta V'$ of the phase L_1 is negative in each of the cases a , b and c , in accordance with (19) curve $(L_1) = F + L_2 + G$ is going starting from point i towards lower pressures (figs 3 and 4).

In the cases a and b the coefficient $(1+a)\Delta H'$ of phase L_1 is positive in equation (20) so that curve (L_1) is going, starting from i , towards lower pressures (fig. 3). In the case c is $(1+a)\Delta H'$ negative and curve (L_1) is going, therefore, starting from i , towards higher pressures (fig. 4). This is in accordance also with that which follows from (18) viz.

$$\left(\frac{dP}{dT}\right)_{L_1} = \frac{\Delta H'}{\Delta V'}$$

Consequently we have defined the direction of the curves (G) and (L_1) ; fig. 3 is true for the cases a and b , fig. 4 for the case c .

With the aid of (19) and (20) we should be able to determine also the position of the curves (F) and (L_2) and then we could prove that the four curves are situated with respect to one another as in figs 3 and 4. [Compare f. i. Communication XIII]. As we know, however, the situation of the curves (G) and (L_1) we can find the position of curves (F) and (L_2) much more easily by using the rule for the position of the four monovariant curves of a binary equilibrium [Compare Communication I fig. 2].

In accordance with this rule we must meet, when we go, starting

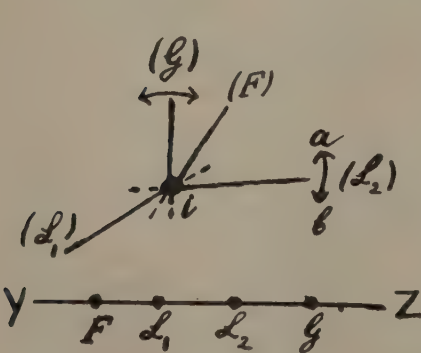


Fig. 3.

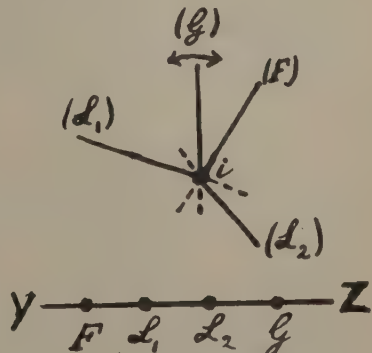


Fig. 4.

from curve (G) in the direction of the hands of a clock towards curve (L_1) firstly curve (F) and afterwards curve (L_2). As further (G) and (F) must form a bundle and their prolongations must be situated between (L_1) and (L_2) and as the angle between two succeeding curves, must be always smaller than 180° , hence follows for the curves (F) and (L_2) a situation as in the figures 3 and 4.

In fig. 3 curve (L_2) is drawn horizontally; starting from i it may run either upwards or downwards; this has been indicated by the double little arrow. When it goes upwards, starting from i , then its prolongation must yet always be situated above curve (L_1). It appears from the coefficient of the phase L_2 in reaction (20) that curve (L_2) must go in case a starting from i upwards and in case b , starting from i downwards. This has also been indicated in fig. 3.

As we know the P, T -diagram of the equilibrium $E(x=0)$ we can easily determine the situation of curve E . It follows viz. from our general considerations in the beginning of this communication, that curve E must be situated between the curves (L_1) and (L_2). For $x_2 : x_1 = \infty$ curve E coincides with (L_2) for $x_2 : x_1 = 0$ with curve (L_1). When $x_2 : x_1$ changes from ∞ towards 0 then curve E moves in the direction of the hands of a clock from (L_2) towards (L_1).

Firstly we now take the case a , so that we must imagine in fig. 3 curve (L_2) to be drawn upwards starting from i . When we do change now $x_2 : x_1$ from ∞ to 0, then it follows from the different positions which curve E may obtain, that the following cases may occur:

$$\begin{array}{lll} (dT)_x > 0 & \text{and} & (dP)_x > 0 \\ (dT)_x > 0 & \text{and} & (dP)_x < 0 \\ (dT)_x < 0 & \text{and} & (dP)_x < 0 \end{array}$$

In case b we must imagine in fig. 3 curve (L_2) to be drawn downwards starting from i . When we do change $x_2 : x_1$ from ∞ to 0, then it follows from the situation of curve E :

$$\begin{array}{lll} (dT)_x > 0 & \text{and} & (dP)_x < 0 \\ (dT)_x < 0 & \text{and} & (dP)_x < 0 \end{array}$$

In case c fig. 4 is true. When $x_2 : x_1$ changes again from ∞ to 0, then it follows from the position of curve E :

$$\begin{array}{lll} (dT)_x > 0 & \text{and} & (dP)_x < 0 \\ (dT)_x < 0 & \text{and} & (dP)_x < 0 \\ (dT)_x < 0 & \text{and} & (dP)_x > 0 \end{array}$$

We see that those deductions are in accordance with the previous ones and with the figs 1 and 2.

Our previous considerations are all valid in the supposition that the four phases F , L_1 , L_2 , and G are situated with respect to one another as is indicated in the figs 1—4. When the four phases are situated otherwise with respect to one another, the reader may deduce all in similar way.

We now shall assume that the new substance is volatile, so that it occurs in the phases L_1 , L_2 , and G with the concentrations x_1 , x_2 , and x_3 .

We find with the aid of (7) and (19):

$$M(dT)_x = -(1+a)\Delta V'_1 x_1 + [a\Delta V'_1 - (1+b)\Delta V]x_2 + b\Delta V \cdot x_3 \quad (36)$$

and with the aid of (8) and (20):

$$M(dP)_x = -(1+a)\Delta H'_1 x_1 + [a\Delta H'_1 - (1+b)\Delta H]x_2 + b\Delta H \cdot x_3 \quad (37)$$

wherein

$$M = (\Delta H \cdot \Delta V' - \Delta H' \cdot \Delta V) : RT$$

so that the direction of temperature and pressure of curve E are defined by (36) and (37).

As ΔV is very small in comparison with $\Delta V'$ we may neglect in (36) the terms with ΔV as long as x_3 is not very large, then it follows with approximation:

$$(dT)_x \gtrless 0 \quad \text{voor} \quad \frac{x_2}{x_1} \gtrless \frac{y-y_2}{y-y_1} \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Only for very great values of x_3 in comparison with x_1 and x_2 , the term $b\Delta V \cdot x_3$ in (36) will be of great importance and will be approximately

$$(dT)_x = \frac{RT \Delta V}{\Delta H \cdot \Delta V'} b x_3 = \frac{RT}{\Delta V'} \cdot \left(\frac{dT}{dP} \right)_G b x_3 \quad . \quad . \quad . \quad (39)$$

In (37) ΔH is not small in comparison with $\Delta H'$ and the term $b\Delta H \cdot x_3$ will assert its influence already with values of x_3 which are not too small.

Consequently, in general the influence of the new substance on $(dT)_x$ and $(dP)_x$ will be larger in proportion as the new substance is more volatile and it will assert its influence sooner on $(dP)_x$ than on $(dT)_x$.

We may also deduce anything about the position of curve E with the aid of the general considerations at the beginning of this communication. Hence it follows viz that curve E must be situated either between the curves (L_1) and (L_2) or between (L_1) and (G) or between (L_2) and (G) . As in the figs 3 and 4 the prolongation of each of those curves is situated between both the other curves, curve E may go, therefore, starting from point i in every direction.

Consequently the temperature may as well increase as decrease, and the pressure may increase or decrease as well at rising as at falling temperature, dependent on the position of curve E .

It follows from (12):

when x_1 is extremely small with respect to x_2 and x_3 , then curve E is situated between (G) and (L_2) ;

when x_2 is extremely small with respect to x_1 and x_3 , then curve E is situated between (G) and (L_1) ;

when x_3 is extremely small with respect to x_1 and x_2 , then curve E is situated between (L_1) and (L_2) ;

when x_1 is extremely large with respect to x_2 and x_3 , then curve E is situated in the vicinity of (L_1) ;

when x_2 is extremely large with respect to x_1 and x_3 , then curve E is situated in the vicinity of (L_2) ;

when x_3 is extremely large with respect to x_1 and x_2 , then curve E is situated in the vicinity of (G) .

In each of those cases we can see at once from the figs 3 and 4 which signs $(dT)_x$ and $(dP)_x$ may have.

When f.i. x_2 is very small with respect to x_1 and x_3 , then curve E is situated between (L_1) and (G) ; when now fig. 4 is valid then the pressure shall, therefore, always increase and the temperature shall decrease. In the special case only, when x_3 is still also extremely large with respect to x_1 and when at the same time $\Delta V > 0$ [then curve (G) proceeds, starting from i , a little to the left] then the temperature may fall a little.

When we add a new substance which is not volatile, but which forms mixed crystals with the solid substance F , then we have in figs. 3 and 4 the curves (F') (L_1) and (L_2) . It appears from the position of those curves with respect to one another that the previous considerations are also valid in this case.

When we wish to calculate $(dT)_x$ then, as is apparent from (19) we have to substitute in (36) $b \Delta V x_3$ by $(\Delta V + \Delta V') x$. When we neglect again the terms with ΔV then we find:

$$M(dT)_x = [x - (1+a)x_1 + ax_2] \Delta V'$$

or:

$$(dT)_x = \frac{RT}{\Delta H} \cdot \frac{x(y_1 - y_2) - (y - y_2)x_1 + (y - y_1)x_2}{y_1 - y_2} \quad . \quad . \quad (40)$$

In the figs 5 and 6 YZ represents a side of the components-triangle, $F L_1 L_2$ and G the four phases of the invariant binary equilibrium $E(x=0)$. When we add a new substance then the ternary equilibrium $E = F + L_1 + L_2 + G$ arises. The solid sub-

stance F and the liquids L_1 and L_2 , then proceed along the curves Fqr , $L_1q_1r_1$ and $L_2q_2r_2$. When we add only little of the new substance, then the 3 phases are represented by the points q , q_1 and q_2 , which we must imagine in the immediate vicinity of the side YZ .

When we put $t = x(y_1 - y_2) - (y - y_2)x_1 + (y - y_1)x_2$, and when we consider x and y as running coordinates, then $t = 0$ represents the equation of the straight line which goes in fig. 5 and 6 through q_1 and q_2 .

When the point q is situated on the line q_1q_2 , then $t = 0$; the sign of $(dT)_x$ is then determined by the terms which have been neglected in (40).

When q is situated at the right side of the line q_1q_2 (viz. when we go from q_2 towards q_1) as in fig. 5, then $t > 0$; when q is situated at the left side of the line q_1q_2 , as in fig. 6, then $t < 0$. Hence it follows, therefore, that in fig. 5 the temperature increases and in fig. 6 the temperature decreases, as is also indicated in both figures.

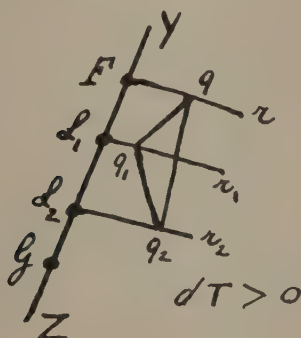


Fig. 5.

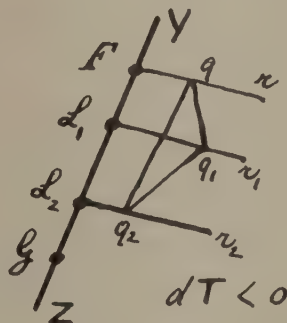


Fig. 6.

Consequently we find the following:

when we add to the invariant binary equilibrium $E(x=0) = F + L_1 + L_2 + G$ a substance which is not volatile and which forms mixed crystals with the solid substance F , then

the temperature rises, when the threephases-triangle solid-liquid liquid turns its concentrated liquid towards the side of the components-triangle (fig. 5)

the temperature falls when the threephases-triangle turns its concentrated solution away from this side (fig. 6).

Comparing fig. 1 with fig. 5 and fig. 2 with fig. 6, the reader will see that for the change of temperature the same rules are true, independent of the fact whether the new substance forms mixed crystals with F or not.

Finally we could still treat the general case that the new substance forms not only mixed crystals with F but that it is volatile also.

It follows from figs. 3 and 4, in connection with the theories discussed in the beginning of this communication that curve E can go in all directions, starting from point i .

In order to define $(dT')_x$ we must still include in (36) the term $(\Delta V + \Delta V')_x$; then we get again (40) approximately unless x , is extremely large.

Consequently in this case also the figs. 5 and 6 remain valid, unless the threephases-triangle $q q_1 q_2$ becomes very narrow and the concentration of the new substance in the vapour is extremely large.

(To be continued).

Leiden, *Inorg. Chem. Lab.*

Anatomy. — "*The Development of the Shoulder-blade in Man*".

By O. H. DIJKSTRA. (Communicated by Prof. L. BOLK).

(Communicated at the meeting of March 24, 1923).

Unlike the development of the clavicle that of the scapula has received comparatively little attention. The textbooks of anatomy (CUNNINGHAM, GEGENBAUER, RAUBER—KOPSCH, MERKEL, POIRIER—CHARPY, TESTUT) contain only general notions such as the information that the ossification of the shoulder-blade begins in the vicinity of the collum scapulae at the end of the second or in the beginning of the third month. POIRIER and CHARPY speak of an incipient ossification between the 40th and 50th day. BARDELEBEN reports a periosteal ossification (such as occurs with the bones of the cranial vault) beside and under the spina scapulae at the end of the 10th week.

BRYCE alone enters into more details in QUAIN'S *Elements of Anatomy*. According to his description the rudiment of the shoulder-blade is in the 6th week entirely cartilaginous, proc. acromialis and proc. coracoideus are present, but the spina scapulae is wanting. (Nevertheless BRYCE reproduces the diagram of LEWIS¹⁾, in which a spina is really indicated). In the 8th week ossification begins with a centre near the collum scapulae, developing into a triangular plate, at whose upper margin the spina appears in the 3rd month as a low ridge. At birth coracoid and acromion, margo vertebralis and the margin of the spina are still made up of cartilage. This description by BRYCE agrees fairly well with the one we find in BROMAN'S textbook of Embryology and in that of KEIBEL and MALL, in which BARDEEN deals with this subject. BROMAN, like BRYCE, states that no spina is to be found at the cartilaginous scapula. Nonetheless he reproduces the figure of LEWIS, in which there is indeed a spina. KOLLMANN, SCHENCK, MINOT, PARKER do not speak of the first development of the shoulder-blade and only dwell on stadia of advanced ossification. In HERTWIG'S *Entwicklungsgeschichte* BRAUS and also HERTWIG himself report a separate centre of ossification in the spina scapulae; according to the latter the spina in the neonatus still consists of cartilage sometimes; according to KÖLLIKER (quoted by BADE, *Arch. f. mikr. Anat.* LV) this is even always the case.

¹⁾ *Am. Journ. Anat.* Vol. I. 1901—'02.

The most detailed report concerning the development of the shoulder-blade is that by BRYCE and BROMAN. From their figures it is evident that they derive their data from LEWIS, who published in the *American Journal of Anatomy* (Vol. I 1901—'02) a minute description of the development of the arm in man. Broadly stated his data agree with those of BRYCE, mentioned above. They differ, however, as to the spina scapulae. According to LEWIS the spina probably takes origin in the upper margin of the scapula. This margo superior thickens and then splits into a medial and a lateral lip. The medial lip is the future margo superior, the lateral one is the first beginning of the spina scapulae.

HAGEN¹⁾ describes a shoulder-blade of an embryo 17 mm. in length. The spina scapulae is absent, the proc. coracoïdeus is large, the proc. acromialis small. The latter statement cannot be reconciled with LEWIS's communication, which, on the contrary, speaks of a relatively large proc. acromialis.

This review of the literature would not be complete without mentioning the interesting study by RUTHERFORD²⁾ who entered into many details of the development of the shoulder-blade. Like LEWIS he constructed wax models of the skeleton of the shoulder-girdle, and i. a. found that the spina scapulae originates in very early ossification of derivatives of cartilage cells, situated between M. supra- and infraspinatus.

From this review it is clear that our knowledge of the modus of development of the shoulder-blade in man is still limited. The shape in the initial stages of development is described differently. Conflicting views are held as to the genesis of the spina and from the contents of this paper it will be seen that these are not the only points of controversy.

With a view to trace the development of the shoulder-blade in man, I constructed wax models of various stages of development. Fig. 1 represents the wax model of the shoulder-blade of the youngest embryo, 16 mm. in length. The scapula is drawn from the lateral side and from above.

The reconstruction shows:

1°. that the shoulder-blade lies in a sagittal plane, so that the lower half is in contact with the three upper ribs. Processus acromialis and clavicula are not in contact as yet.

2°. that the processus coracoïdeus is large; the processus acromialis is relatively small. The joint-cavity rests chiefly on the processus coracoïdeus.

¹⁾ *Arch. f. Anat. u. Entwickel. Gesch.* 1900.

²⁾ *Journal of Anatomy and Physiology* 1914.

3°. There is no indication of a spina scapulae. The margo superior is neither thickened nor split into two labia.



Fig. 1.



Fig. 2.

4°. The margo superior is straight, so there is no incisura scapulae.

5°. For the rest the shape of the scapula fairly well agrees with that of an adult shoulder-blade. In reconstructing the scapulae of two monkey embryos (viz. *Macacus cynomolgus* 17 mm. in length, and *Semnopithecus maurus*) it became evident that, also in these primates, the embryonic shoulder-blade already in its first beginning resembles that of the adult. Here also a spina was absent.

6°. Close beneath the angulus superior we observe a well-defined fovea where a foramen is found in older stages of development. To this we shall revert when discussing the following stage.

This stage is illustrated in fig. 2. It concerns the shoulder-blade of an embryo, 25 mm. in length. Also in this stage any indication of a spina scapulae or of a thickening of the margo superior is lacking. Nevertheless when compared with the first stage some modifications can be recognized.

1°. The shoulder-blade does not lie any more in a sagittal plane, but makes an angle with it, as is also the case with the adult. The joint-cavity lies at the level of the first rib. Acromion and clavicle have joined.

2°. The processus coracoïdeus has comparatively decreased, the processus acromialis, on the other hand, has increased. It appears, then, that the processus coracoïdeus, which is phylogenetically the oldest part, is most strongly developed in the youngest stage, whereas the processus acromialis, which is phylogenetically younger, comes more to the fore in the older stages. The joint-cavity now lies for the greater part on the planum scapulae.

3°. The margo vertebralis consists of a shorter upper portion and a longer lower portion. They are at an obtuse angle to each other.

4°. The portion of the scapula from which afterwards the fossa supraspinata develops, makes an angle with the future subspinal portion. This deviation of the upper part, which also occurs in the adult shoulder-blade (since fossa supra- and infraspinata do not lie in one and the same plane), had not yet taken place in the 16 mm. embryo.

5°. In the cranial part of the shoulder blade a foramen occurs under the angulus superior, which extends at the costal plane of the scapula as a groove along the margo superior in the direction of the joint-cavity. In fig. 3 we give a cross-section of this foramen, which is filled with connective tissue.

The existence of this foramen is no doubt surprising; yet it was not entirely unknown, as already RUTHERFORD has described it (l. c.). However, according to this author it proceeds in a groove, which reaches as far as the margo vertebralis. Now, in all the serial sections in which I also met with a groove as well as with the foramen, it proceeded along the margo superior in the direction of the joint-cavity.

RUTHERFORD explains this foramen as follows. He considers the part of the scapula, cranial to the foramen (resp. groove), as a separate piece of cartilage, which he terms *praescapula*, and which, according to his account, is connected by a strand of mesenchyma tissue with the sternal half of the clavicle. In this way he believes an inner shoulder-girdle to have developed, while he supposes the acromion-clavicle to build up the outer girdle. He adduces various arguments to prove this; however, they are weak. In my judgment the hypothesis is of no value, because a connection of the so-called *praescapula* with the sternal half of the clavicle does not occur. At all events in my preparations I never found a cell-strand like the one described by RUTHERFORD.

This foramen is not present in all cases. Its development also differs with various individuals, as shown by the following data. I could establish its presence either as a true foramen, or as a deep groove in human embryos of the length of 16, 17.5, 18, 19.6, 21, 22, 25 (see fig. 3), 26, 27, 56, and 90 mm. On the other hand I did not

recognize it in embryos of 12, 18, 18, 24, 26, 40, 120 mm. From this it follows that it is not infrequently absent. In some embryos the portion of the planum scapulae cranial to the foramen, i.e. RUTHERFORD's praescapula, made an angle with the rest of the planum, a fact that lends support to RUTHERFORD's view, viz. that it is really a separate piece

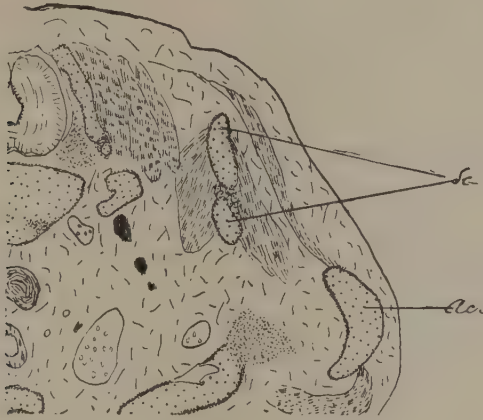


Fig. 3. Homo 25 mm. transverse. *Sc* = Scapula;
Acr = Processus acromialis.

of cartilage. The foramen which, in young embryos, is situated rather closely to the margo superior, as observable in fig. 3, migrates in older embryos towards the margo vertebralis. Consequently RUTHERFORD's praescapula is relatively enlarged.

Now it is an interesting fact that this foramen does not occur in any other mammal, neither in reptiles, nor in amphibians. At least I never detected any. The following embryos I have examined for the occurrence of this foramen.

Semnopithecus maurus 20 mm. (C. R.)

Macacus cynomolgus 17 mm. (C. R.)

Cercopithecus 2 stages.

Sus scrofa N. T. (Keibel) 83—85, N. T. 88, N. T. 88, N. T. 91, 24 mm. (C. R.) 26 mm. (C. R.) In the last two embryos two foramina were recognized in the fossa infraspinata. It is not quite impossible that these foramina are analoga of the foramen in the human shoulder-blade.

Bos taurus 21 mm. (C. R.)

Ovis aries 19.5, 20.5, 21.5, 22.5, 23, 23.5, 26, 27, 29, 35, 45 mm. (C. R.)

- Canis familiaris* 12, 12, 22, 23.5 mm. (C. R.)
Sciurus vulgaris 12, 30 mm. (C. R.)
Mus decumanus 11.5, 12, 13, 13, 13.2, 14.5, 16, 18, 20, 22 mm. (C. R.)
Lepus cuniculus 17, 20 mm. (C. R.)
Spermophilus citellus 15 mm. (C. R.)
Roussettus amplexicaudatus 7.5, 10.5, 11, 11, 11.5, 12, 12, 14.5, 15.5, 16, 18 mm. (C. R.)
Talpa europea 8.5, 9, 9, 10, 12, 13, 16.5, 20 mm. (C. R.)
Perameles obesula 50 mm. (C. R.)
Perameles spec. 38 mm. (C. R.)
Dasyurus viverrinus 19.6, 33, 36, 40, 53, 63 mm. (C. R.)
Sminthopsis crassicaudatus 13, 25 mm. (C. R.)
Phascalogale pennicillata 37 mm. (C. R.)
Trichosurus vulpecula 32 mm. (C. R.)
Didelphys cancrivora, 4 embryos of 25 mm. length.
Lacerta agilis N. T. (Keibel) 117, 118, 120, 123, 123, 124, 125, 126.
Calotes iubatus, length of the head $5\frac{1}{2}$ mm., 7 mm.
Lagysoma 27.5 mm.
Hemidactylus fren. length of the head 4.5 mm.
Salamandra mac. 11, 13, 15, 16, 16, 24 mm.
Pipa Americana, 12 mm.
Rana . 2 embryos.

So far as I am able to judge foramina in adult shoulder-blades occur only with *Homo* and with various *Edentata*, in which they are always formed by bridging of the *Incisura scapulae*, and with *Delphinus delphis*. In the latter the character of the foramen is not known. RUTHERFORD (l. c.) has described it.

A conceivable connection, that might exist between the *praescapula* of RUTHERFORD and the attachment of the *clavicula* (not only the sternal half of the *clavicula*, as RUTHERFORD supposed) to the *margo superior scapulae*, as it occurs in reptiles, *echidna* and *ornithorynchus*, could not be ascertained, since a connection of the *praescapula* of RUTHERFORD to the acromial part of the *clavicula* could not be detected either.

It appears, then, that the foramen, present in the majority of human embryos in the cranial part of the shoulder-blade, does not occur in other vertebrates, (except in *Delphinus delphis*, which, however, is of such a pronounced specificity that this foramen cannot be looked upon as a homologue of that of man). Neither did I find any attachment of the *praescapula* of RUTHERFORD to any other

skeletal bone. The significance of this foramen is unknown as yet.

As to the ossification of the scapula my experience proved it not to be so simple as is represented in the literature.

The earliest ossification I observed in an embryo of 40 mm. I constructed a wax model (fig. 4) of the scapula of this embryo.



Fig. 4.

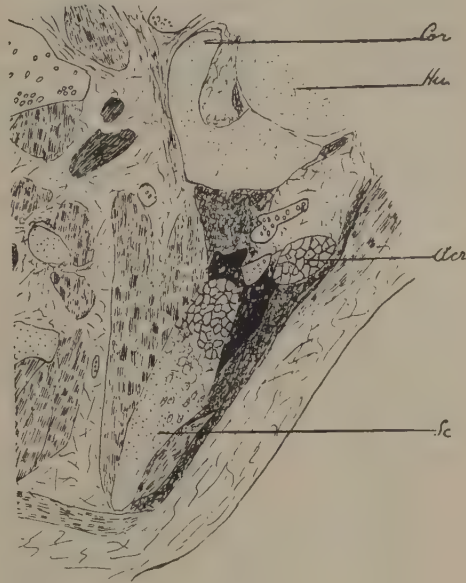


Fig. 5. Homo 40 mm. Transversal.

Cor = Processus coracoideus; Hu = Humerus;
Acr = Processus acromialis; Sc = Scapula.

Like the preceding model this also is viewed from above and from the dorso-lateral side. What this reconstructed model shows us may follow here:

The joint-cavity, lying at the level of the first rib, is now located almost entirely on the planum scapulae (as with the adult scapula). Of the spina not a trace is visible as yet, the margo superior is not thickened. To the basis of the processus acromialis an area of closely packed mesenchyma is attached, which extends between the muscular tissue and separates the rudiment of *Musc. supra-*, and *infraspinatus*.

This area of mesenchyma is cut in a cross section as represented in fig. 5. Behind the root of the processus acromialis begins a perichondrial ossification, which continues into this condensed mesenchyma. This ossification is the first formation of the spina. We see,

therefore, that it is formed by a perichondrial ossification, for although no ossifying perichondrium is visible here, the fact that the bone is formed from the surrounding mesenchyma co-ossifying with cartilage, established the character of the ossification. In fig. 5 we give a cross section of this first stage of the spina.

I have not been able to recognize two centres of ossification in the cartilaginous scapula, described by RAMBAUD and RENALT (quoted by POIRIER¹), which, according to these authors, arise between the 40th and 50th day and fuse in the third month.

In the scapula of an older embryo (56 mm. in length) this perichondrial ossification appears to be largely extended. The margo anterior scapulae is almost reached. The cartilage of the planum

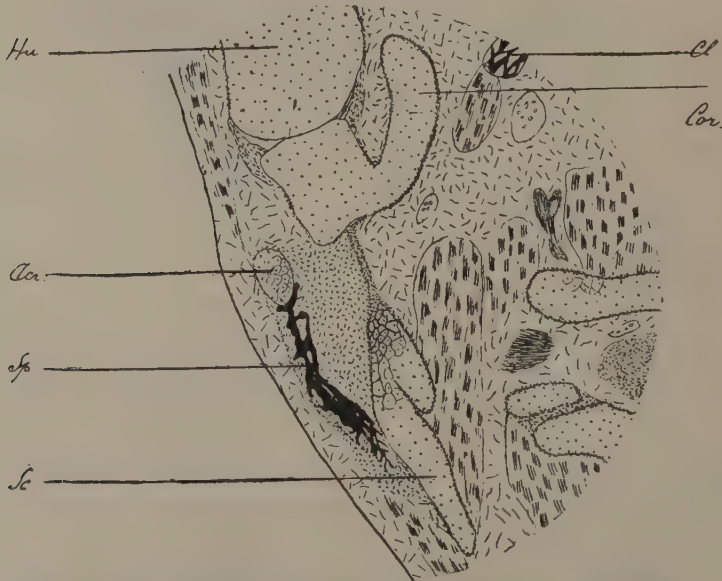


Fig. 6. Homo 56 mm. Transversal. *Hu* = Humerus; *Cl* = Clavicle;
Cor = Processus coracoideus; *Acr* = Processus acromialis;
Sp = Spina scapulae; *Sc* = Scapula.

scapulae, however, has been distinctly calcified over a considerable area already. The marked enlargement of the spina scapulae is shown in fig. 6. Besides the spina this figure also shows part of the foramen described above. The spina is formed by a growth of bone between

¹) POIRIER et CHARPY, *Traité d'Anatomie humaine*.

M. supra- and infraspinatus, between acromion and planum scapulae. It cannot be denied, however, that in the mesenchyma, in which this bone develops, very young cartilage-cells are noticeable here and there. These cells, however, have no intermediate matter as yet; they are little differentiated and it is difficult to distinguish them from the mesenchyma-cells. So it is evident that besides bone-cells also cartilage-cells develop in the mesenchyma.

In an embryo of 90 mm. enchondrial as well as perichondrial ossification takes place, the boundary between the two being no



Fig 7. Homo 90 mm. Margo anterior scapulae transversal.



Fig. 8. Homo 90 mm. Scapula transversal *Acr.* = Processus acromialis
J.c. = Joint-cavity. *Pl. Sc.* = Planum scapulae.

longer perceivable. The peculiar character of the perichondrial ossification along the margo anterior is remarkable. In the place of the formation of compact bone, which in other cases occurs with perichondrial ossification e.g. that of the long bones, we see here a bony framework encircled by mesenchyma. Fig. 7 shows a cross section through the margo anterior.

The study of this object (embryo of 90 mm.) shows remarkable peculiarities of the growth of the spina scapulae. In the mesenchyma between M. supra- and infraspinatus a distinct cartilage is now recognizable. It is quite independent of the other mass of cartilage

of the scapula. It is younger than the remaining part of the shoulder-blade; nevertheless it has already calcified to some degree and forms bone of the spina.

The cartilage has been cut in three different cross sections, as represented in the figures 8, 9 and 10. Fig. 8 illustrates a section through the scapula above the place of attachment of the processus acromialis. In the mesenchyma, which extends from the processus acromialis towards the margo vertebralis, lies the cartilage which is already partly calcified. In fig. 9 we give a section at a lower level.

The processus acromialis attaches itself at this level to the planum scapulae. Here also we observe the cartilage of the spine, independent of the remaining cartilage of the shoulder-blade. Fig. 10 shows a section through the scapula at the level of the lowest place of



Fig. 9. Homo 90 m.m. Scapula transversal. *Acr.* = Processus acromialis.
Pl. Sc. = Planum scapulae.



Pl. Sc.
Fig. 10. 90 m.m. Scapula transversal. *C.* = cartilage of the spine. *Pl. Sc.* = Planum scapulae.

attachment of the spina. The young cartilage, which forms the spina, has here been cut over a large area. The cartilage will be seen to

be partly calcified, while bone has been formed, uniting with this calcified area.

So while the first beginning of the spina is formed by perichondrial bone in the mesenchyma between *M. supra-*, and *infraspinatus*, its further development is effected by chondrial bone, which originates in the younger cartilage. This cartilage has been generated between the afore-said muscles by the same mesenchyma.

A peculiar feature is still to be observed at the shoulder-blade of the embryo of 90 mm. Bone is developed at the margo superior as well enchondrially as perichondrially. In the mesenchyma that forms the perichondrial bone, and into which this bone extends over some distance, there are two cartilaginous nuclei, made up of the same young tissue from which the cartilage of the spina has been built up. Fig. 11 shows in cross section these nuclei, which are not in contact with the remaining cartilage of the shoulder-blade. These cartilage-islets appear to be already calcified and ossified here and there. It is impossible to draw a boundary-line between the bone formed in this process and the perichondrial bone of the scapula. This ossifying process, in which (besides the enchondrial ossification of the scapula) both perichondrial and chondrial ossification of a cartilage nucleus, situated outside the perichondrial bone, are present, agrees completely with the formation of the spina scapulae. This is striking, since the spina scapulae and the definitive margo superior are the two parts of the shoulder-blade, which are missing in the first rudiment of the cartilaginous scapula. This deficiency vertebral of the place destined for the future incisure, is indeed accounted for by the fact that the margo superior in young embryos is still straight and displays no incisure. The missing parts are apparently supplied by the perichondrial bone that reaches far into the mesenchyma, together with the bone formed by the afore-said cartilage-nuclei. At the shoulder-blade of an embryo of 120 mm. in length, in which the ossification had considerably advanced, the incisure was indeed present.

Of course, the question arises, how the cartilage of the spina as well as the cartilage nuclei are further developing. In both places the cartilage is soon transformed completely into bone. In an embryo of 120 mm. only a very few remnants of the cartilage of the spina were still left. The rest had been ossified.

After this the development of the shoulder-blade proceeds in the way described in the text-books of embryology.

Now let us review once more the current opinions of the development of the spina scapulae. It will be seen, then, that however

divergent they may be, most of them cannot be deemed incorrect, when we bear in mind that they concern different stages.



Fig. 11.
Homo 90 m.m.
Margo superior scapulae
transversal.

RUTHERFORD's view of the very early ossification of cartilaginous cells is no doubt correct, but holds good only for young stadia. Neither is the conception of HERTWIG and BRAUS about a separate centre of ossification quite incorrect, since there is a stage in which an independent cartilage is forming bone. BARDELEBEN's record about an ossification under and beside the spina cannot altogether be disqualified either, but it only applies to a brief stage of development. However, ossification like that of the bones of the cranial vault does not occur in the development of the shoulder-blade. In the neonatus a few cartilage may possibly sometimes be found at the spina (BRYCE), but it is certain that the spina scapulae in the new-born child does not consist of cartilage. (KÖLLIKER and HERTWIG advocate the opposite view). LEWIS's conception, however, (doubling of the margo superior) is altogether wrong. The diagram borrowed from LEWIS by BROMAN, BRYCE and BARDEEN represents a faulty reconstruction of the shoulder-blade.

Zoology. — “*Secondary sex-characters and testis of the ten-spined Stickleback (Gasterosteus pungitius L.)*.” By Dr. G. J. VAN OORDT. (Communicated by Prof. J. BOEKE).

(Communicated at the meeting of March 24, 1923).

It is generally known that the sex-glands strongly influence the so-called secondary sex-characters. This is apparent from the marked somatic and psychic differences, which e. g. Mammals or Birds, castrated at an early age, show, when compared with normal animals.

At present it is generally accepted that in Vertebrates this effect, resulting from the gonads, takes place by internal secretion, that is by the influence of certain substances, which pass into the blood (“hormones”). As the correlation between the secondary sex-characters and the gonads generally is most distinct in male Vertebrates, I will speak only of the formation of these hormones in the testis for convenience’ sake.

Recently it has been especially attempted to ascertain, by which part of the male gonad these hormones are formed. The numerous investigators, treating this subject, chiefly hold the two following, contradictory opinions.

According to STIEVE (1922) and others these hormones are exclusively formed by the sexual cells, whereas BOUIN and ANCEL (1903), STEINACH (1920), LIPSCHÜTZ (1919), BASCOM (1923), their collaborators and others are of opinion that these hormones originate in the interstitial cells (LEYDIG’s cells), situated in the interstitium of the male gonad. According to STIEVE these cells are only thropic elements for the sperm cells. Consequently no value must be attached to the name „Puberty Gland”, which name was given to the collective LEYDIG’s cells by STEINACH and LIPSCHÜTZ.

Up till now the investigators, when treating the subject above mentioned, have chiefly examined Mammals, Birds and Amphibia. For that reason I resolved to trace the changes in the testis at the appearance of the secondary sex-characters in a Fish, and so I chose the ten-spined Stickleback (*Gasterosteus pungitius* L.), which was easy to obtain.

During breeding time, in spring, the males of this species possess

a number of secondary sex-characters (cf. TITSCHACK 1922), of which the following are distinctly perceptible.

In spring a very distinct black pigmentation (red in the three-spined species) can be observed at the throat and at the abdomen, which soon spreads over the rest of the body, so that the animals become dark-black, except for their pectoral spines. Outside breeding-time it is difficult to distinguish the males from the females: then both show dark spots on a pale green ground. Individual colour-differences occur.

Every male makes a nest, in which the eggs are deposited. The material of which the nest consists (parts of waterplants etc.) is collected by the male and fastened by means of a secretion, formed by the kidney-tubules and Wolffian Ducts (TITSCHACK 1922, COURRIER 1922*b*, both in *Gasterosteus aculeatus* L.). This peculiar secretion occurs exclusively in the male during breeding time; for that reason in spring the kidney strongly increases in size, the kidney-tubules and the Wolffian Ducts get a larger diameter and exercise a different function.

The male guards his nest and drives off all intruders fiercely. When the eggs have been deposited in the nest, they are at once fertilized. During the development of the eggs, the male takes care that they are constantly provided with oxygen by conducting fresh water to the nest with his pectoral fins. Sometimes, when eggs drop out of the nest, they are again collected by the male and taken back to the nest in his mouth. Whether the young are guarded by the male, after they have left the nest, in nature, is not known to me: care must be taken to separate the young, living in prison, from their father and the other inhabitants of the aquarium, as the young will otherwise be eaten.

The aim of my investigation, begun in September 1922, was to trace the changes, occurring in the testes of the Stickleback at the appearance of the secondary sex-characters. So it was my intention to catch a number of Sticklebacks at fixed times during the winter and the succeeding spring and to examine their sex-glands. At that time I thought that nothing was known as yet about the relation between the secondary sex-characters and the testis of the Stickleback, but it soon appeared to me that COURRIER had already investigated the three-spined Stickleback (*Gasterosteus aculeatus* L.) and had published some papers, regarding this point (1922*a*, 1922*b*).

I therefore changed my original plan and resolved to trace what influence a rather high temperature, about the temperature of

ditchwater in spring (12° — 20° C.), would have on the appearance of the secondary sex-characters and what changes would take place in the testes of these animals simultaneously. The sex-glands of control-animals, caught in nature, could serve at the same time to verify the results of COURRIER. In this paper I will only communicate the results, obtained in animals, kept in a temperature of 12° — 20° C. during last winter.

In September and October 1922 I caught a large number of specimens of *Gasterosteus pungitius* L. at Rotterdam. They were kept in an aquarium of which the water was often renewed, and they were copiously fed with *Chironomus*-larvae.

All the testes of the Sticklebacks, killed in autumn, contained a more or less large number of spermatozoa. The number of spermatogonia is always small, the number of spermatocytes and spermatids varies in the different specimens. In all cases, examined by me, small groups of interstitial cells (LEYDIG's cells) were present, close to the hilus or there where three or more tubules come together. In a few testes, in which the interstitium is somewhat wider, these cells are also situated between the seminiferous tubules. They were absent in none of the cases examined.

In one specimen (n^o. 6), a rather dark-coloured male, not showing the black pigmentation of males during breeding time, however, the interstitium is much wider than in the other males, caught at the same time. The number of interstitial cells is also larger in this specimen, while in the seminiferous tubules spermatozoa are almost exclusively found.

Oblong connective tissue-nuclei are observed everywhere in the interstitium of the testes of animals, caught in autumn; blood-vessels are present, but they are not numerous; they are narrow and contain few blood-cells.

This testis-structure is shown by animals, caught in September and the beginning of October, and which were kept in an aquarium of which the water then agreed in temperature with ditchwater.

The testes of Sticklebacks, kept for two, three and even four months, i. e. till the end of January 1923, in a temperature of 12° — 20° C., all increase in size and show the following structure. The spermatogenesis is very intensive. In all testes this process takes place from the exterior to the interior, i. e. the spermatozoa are situated as a rule more in the centre, the spermatogonia and spermatocytes more at the periphery of the gonad. The interstitium of such animals does not change; it remains narrow, the number

of LEYDIG's cells is generally small and they are especially present near the hilus and there where three or more seminiferous tubules come together.

Till the end of January it was difficult to distinguish the males from the females. In the last days of January, however, one of my specimens showed at throat and abdomen a faint black pigmentation, which soon increased strongly. Besides, this animal became very aggressive and in the beginning of February he began to collect material for the nest. On the 14th or 15th of February the eggs were laid in the completed nest; (I cannot give the exact date, as the female was not seen in this nest). On the 16th of February this male was killed.

The nuptial colours successively developed in the other males, which soon began to prepare their nests. After the eggs had been deposited in them, they were carefully guarded by the males, which constantly conducted fresh water to the nests.

On comparing the testes of animals killed in the end of December or in January with the testes of these males, we see that the latter have greatly changed.

The spermatogenesis has totally come to an end. The seminiferous tubules are entirely filled with a large number of spermatozoa. Moreover, at the periphery of the tubules small groups of spermatozoa are to be seen, the heads of which are directed to the wall and the tails to the centre of the tubules. The number of spermatogonia and spermatocytes has strongly decreased.

The interstitium is no longer narrow but is enlarged; the number of LEYDIG's cells has strongly increased; the blood-vessels have become more numerous and larger.

So we see that the high temperature of the water in winter favours the spermatogenesis and that consequently after four months a testis originates of which the seminiferous tubules practically contain spermatozoa exclusively. Then the secondary sex-characters distinctly develop, the interstitium is enlarged and the cells of LEYDIG and the blood-vessels increase in number.

So I have observed a coincidence of the occurrence of the secondary sex-characters and the termination of the spermatogenesis, while simultaneously an enlargement of the interstitium with increase in number of the LEYDIG's cells and of the bloodvessels takes place. This does *not* prove, however, that a correlation exists between these phenomena.

According to COURRIER (1922*a*, 1922*b*) it does. This investigator observed in the three-spined Stickleback that after the spermato-

genesis the interstitium increases considerably in size. In it a strong augmentation of the number of LEYDIG's cells and of the blood-vessels has taken place. According to COURRIER the testes of Sticklebacks, caught in winter, only contain a few interstitial cells here and there. The spermatogenesis, which is very intensive in spring till the end of March, has no influence on the development of the secondary sex-characters. The latter occur not earlier than at the end of April, simultaneously with the strong development of the interstitial cells. As he, moreover, observes the same granules in the cells of LEYDIG and in the bloodvessels, situated close to them, he assumes that the hormones which influence the development of the sex-characters are formed in the interstitial cells and pass from the latter into the blood. In my opinion it might be that the granules, observed by COURRIER, are transmitted by the blood to the interstitial cells.

COURRIER has also kept his fishes in water of 17° C. (1922*a* and 1922*b*, p. 137) during a part of the winter. After two months and a half the structure of the seminiferous tubules of these animals resembles that of animals during breeding time i. e. they are entirely filled with spermatozoa and contain only a few spermatogonia, spermatocytes and cells of SERTOLI. Changes in the interstitium have not occurred. Consequently, the secondary sex-characters have not developed in these animals. COURRIER thinks, however (1922*a*, in a note), on the ground of experiments, which were in progress at that time, that the interstitium would increase in size, when exposed longer to a high temperature and that consequently the sex-characters would also develop in these animals.

I think I am justified to conclude from my investigations, described above, that the correlation of interstitial cells and secondary sex-characters is not so easy to establish.

In the first place *all* testes of *Gasterosteus* possess a more or less large number of interstitial cells. These evidently do not cause the development of the secondary sex-characters. Here I must especially point to the male above described (N^o. 6) of which the testes contain a wide interstitium with many LEYDIG's cells and of which the seminiferous tubules are entirely filled with spermatozoa. The secondary sex-characters had not developed in this animal, however. Among the testes of control-animals, caught in nature in winter, I also found some of which the tubules almost exclusively contained spermatozoa and of which the interstitium with numerous interstitial cells is rather strongly developed. These animals, however, did not show sex-characters either.

In a very recent paper CHAMPY (C. R. Soc. de Biologie, Séance du 17 Février 1923) communicates that he has obtained Sticklebacks (*aculeatus*) with nuptial colours last winter and that in the testes of these animals he had not observed a well-developed interstitial tissue. As he has not found any interstitial cells in the testes of various species of fishes with distinct secondary sex-characters, CHAMPY is of opinion that these cells have no influence on the development of those sex-characters and that the formation of the hormones responsible for the development of these characters would take place by means of the sexual cells.

Finally, I will once more call attention to the fact that the testes, examined by me, in which the spermatogenesis has almost come to an end, possess a more strongly developed interstitium than testes, in which the spermatogenesis is still in full swing. Possibly this fact points to a correlation between spermatogenesis and interstitial cells. Whether the sex-hormones are formed in the seminiferous tubules as well, I cannot decide at this moment. Later on, when I have more material at my disposal, I hope to recur to this subject in a more detailed paper.

Zoölogical Laboratory of the Veterinary College.
Utrecht, March 1923.

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